Environment. Technology. Resources. Rezekne, Latvia
Proceedings of the 15th International Scientific and Practical Conference. Volume II, 143-149

On special finite difference approximations for solving second order differential equations

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Abstract. **The described special methods are applicable for various mathematical physics problems with second-order differential equations involving periodic boundary conditions (PBCs) and first-order homogenous boundary conditions (FBCs). Solutions of some linear and nonlinear problems for parabolic type partial differential equations (PDEs) with FBCs are obtained, using the method of lines (MOL) to approach the PDEs in the time and the discretization in space applying the finite difference scheme with exact spectrum (FDSES). For PBCs we use the finite difference scheme (FDS) for locally approximating periodic function's derivatives in a 2n+1, n>=1 -point stencil, obtaining higher order accuracy approximation. This method in the uniform grid with N mesh points is used to approximate the differential operator of the second and the first-order derivatives in the space. In this paper, we show that the approximation using the FDSES method is equivalent to the spectral differentiation matrix method based on trigonometric (Fourier) interpolant.**
Considering, that the solutions obtained in solving nonlinear problems can be very significantly different from classical solutions, for example, mathematical modelling of processes where temperature or energy is concentrated in a very narrow interval or around a point, again causes increased interest in such areas of application as laser technology, military sphere, etc.
In this regard, also in the given publication, the solution of the "blow-up" phenomenon of the boundary problem of the nonlinear heat conduction equation has been studied and obtained with the above-mentioned high-accuracy solving methods.

Keywords: "Blow-up" phenomenon, differentiation matrices, finite difference scheme with exact spectrum, multi-points stencil, trigonometric interpolant.

1. Introduction

In the last three decades, the concept of a differentiation matrix (DM) to be a very useful tool in the numerical solution of differential equations is developed. DMs are derived from the spectral collocation or pseudo-spectral method for solving differential equations of boundary value type [5], [7], [6], [4], [10]. In the spectral collocation method, the unknown solution is expanded as a global interpolant, such as a trigonometric or polynomial interpolant. The DMs are based on Chebyshev, Fourier, Hermite, and other interpolants.

Spectral DMs for problems with PBCs are based on Fourier interpolant. In other methods, such as finite elements or finite differences, the expansion involves local interpolants such as a piecewise polynomial.

In practice the accuracy of the spectral method is superior- for problems with smooth solutions convergence rates of $O(e^{-cN})$ are achieved ($c = const\ > 0$) in [6], [3], [4]. In contrast, finite elements or finite differences on 3-point stencil yield convergence rates that are only algebraic in N, typicaly $O(N^{-2})$. The spectral collocation method for solving differential equations is based on weighted interpolants of the form [7]:

$$f(x) \approx P_{N-1} = \sum_{j=1}^{N} \frac{\alpha(x)}{\alpha(x_j)} \Phi_j(x) f(x_j).$$

Here $\{x_j\}_{j=1}^{N}$ is set of distinct interpolation nodes (grid points), $\alpha(x) > 0$ is a weight function and the set of interpolation functions $\Phi_j(x_k) = \delta_{jk}$ satisfies $\Phi_j(x_k) = \delta_{jk}$ (the Kronecker delta), $f(x_k) = P_{N-1}(x_k), k = \overline{1, N}$.

For Chebyshev, Hermite and other interpolant the interpolating functions $\Phi_j(x)$ are polynomials of degree $N-1$.

For nonpolynomial cases there are trigonometric interpolants. The collocation derivative operator is generated by taking m - order derivatives of the interpolants and evaluating the result at the nodes $\{x_k\}$. The derivative operator may be represented by matrix $D^{(m)}$ (DM) with entries $D_{kj}^{(m)} = \frac{d^m}{dx^m}\left[\frac{\alpha(x)\Phi_j(x)}{\alpha(x_j)}\right]_{x=x_k}$.

Print ISSN 1691-5402
Online ISSN 2256-070X
https://doi.org/10.17770/etr2024vol2.8020

The numerical differentiation process may therefore be performed as the matrix-vector product
$F^{(m)} = D^{(m)}F$, where $F, F^{(m)}$ are the vectors of function values at the nodes. When solving differential equations, the derivatives are approximated by these discrete derivative operators.
For boundary value problems with the PBCs the DMs $D^{(m)}$ are circulant and there can be given with the first row.

For such matrices it is easy to do arithmetic operations in shorter computation time. Also, it is possible to get the inverse matrix analytically. Orthonormal eigenvectors w_k, w_k^* with the elements
$w_{k,j} = \sqrt{N^{-1}}\exp(2\pi ijk/N)$,
$w_{k,j}^* = \sqrt{N^{-1}}\exp(-2\pi ijk/N)$, $i = \sqrt{-1}, k; j = \overline{1,N}$ do not depend on the elements of circulant matrix.
PBCs allows to freely increase approximation order by increasing the stencil of grid points.
For, example, $2n+1$ points stencil needs to use additional discrete conditions of periodicity
$u_k = u_{N+k}, k = \overline{-n,n}$.
Thus, can be obtained algorithms with higher order precision (different order FDS).

By solving the discrete spectral problem, we can express the matrix $A_{m,n}$ in the form $\mathrm{A_{m,n}} = \mathrm{WD_{m,n}W^*}$, where W is the complex matrix which consists of the eigenvectors in its columns, $\mathrm{W^*}$ is the conjugate transpose of W and $D_{m,n}$ - diagonal matrix with the eigenvalues $\mu_{m,n}$ of matrix $A_{m,n}$ on the diagonal, $m = 1; 2, n = \overline{1,N}$.

Eigenvalues are obtained analytically for every multi-point stencil [8], [10].

Finite difference methods are important for approximating differential operators and solving various ordinary and partial differential equations numerically.

Probably, the most casual is the second-order accurate FDS for approximation of the first and second-order derivatives in a uniform 3-point stencil.

2.MATERIALS AND METHODS

In this chapter, we will consider special difference schemas for solving ordinary and partial differential equations, and special numerical methods (to find the solution of the discrete problem by choosing a suitable transformation or Fourier series elements for finding the components of the discrete problem).

2.1 Special difference schemes for solving differential equations

In this paper, more accurate methods for approximation of the first and second order derivatives in a uniform multi-point $2n+1, n \geq 1$ stencil are investigated. The algebraic convergence rate is $O(N^{-n})$.

We define the FDSES [1], [9] where the finite difference matrix A is represented in the form $A = WDW^*$, W, W^* are the complex and conjugate-complex matrices of finite difference eigenvectors, D is diagonal matrix of the discrete eigenvalues and the elements of diagonal matrix D are replaced with the first N eigenvalues from the differential operator.

In the first publication about FDSES [2] the finite differences with the second order of approximation in the uniform grid are used for the approximation of the second order derivative in the space segment $x \in [0, L]$ with the FBCs.

Special numerical algorithms are developed for solving 1D and 2D problems of the second order ordinary (ODE) and partial differential (PDE) equations with PBCs.

The linear heat transfer equations with variable coefficients can be written in the following form:

$$u_t(x,t) = k(x)\big(u(x,t)\big)_{xx} + p(x)\big(u(x,t)\big)_x + q(x)u(x,t) + f(x), u(x,0) = u_0(x), \quad (1)$$

where $k(x), p(x), q(x), u_0(x)$ are real functions, $x \in (0, L), t > 0$ are the space and time variables, L is the period for PBCs or the length of the segment, $u = u(x,t)$ is the unknown function. For the similar system of PDEs k, p, q are matrices, u is column-vector.

The heat transfer problem is solved numerically using the method of lines and two ways of finite difference methods for the approximation of spatial derivatives - local approximation with finite differences in uniform grid (FDS, FDSES) and global approximation with differentiation matrices.

For local approximation we have the discrete equations ($x_j = jh, Nh = L, j = \overline{1,N}$) as a system of ODEs in following form:
$\dot{U} = \big(KA_{2,n}\big)(U) + \big(PA_{1,n}\big)(U) + Q(U) + F, U(0) = U_0$,
where $A_{1,n}, A_{2,n}$ are N -th order circulant matrices, $U = U(t), F = F(t), \dot{U} = \dot{U}(t), U_0$
are column-vectors of the N-th order with elements $u_j(t), f\big(x_j(t)\big), u_t(x_j,t), u_0(x_j)$,
K, P, Q are N-th order diagonal matrices with elements $k(x_j), p(x_j), q(x_j)$
(in the case of the constant matrices k, p, q we have Kronecker tensor products $k \otimes A_{2,n}$, $p \otimes A_{1,n}$, $q \otimes I$, I is N -order unit matrix).

2.2 Special numerical methods for approximations of derivatives

To determine the special numerical methods the solution of the discrete problem can be obtained with:

a) the transformation $V = WU^*$ by reducing the vector problem to scalar-separated problem with the discrete eigenvalues,
b) the complex discrete Fourier series for vector components $u_j = u(x_j), f_j = f(x_j)$ using the discrete orthonormal eigenvectors w_k, w_k^* and eigenvalues $\mu_{m,k}$ of matrix $A_{m,n}, m = 1,2, k = \overline{1,N}$,
c) the real discrete Fourier series for vector components u_j, f_j using trigonometrical functions $\sin(2\pi kj/N), \cos(2\pi kj/N), k = \overline{1,N/2}$ (in this case the real discrete Fourier expression from matrix $A_{m,n}$ spectral representation is obtained).

For forming complex eigenvalues of FDSES method in the diagonal matrices $D_{m,n}$ elements d_k, the discrete eigenvalues $\mu_{m,k}$ are replaced with the first N eigenvalues $\lambda_{m,k}, m = 1;2$ in special way (N - even):

1) $d_k = \lambda_{2,k}, k = \overline{1,N/2}, d_{k+N/2} = \lambda_{2,N/2-k}, k = \overline{1,N/2-1}, d_N = 0$ (see Fig. 1.),

2) $d_k = \lambda_{1,k}, k = \overline{1,N/2-1}$, $d_{k+N/2} = -\lambda_{1,N/2-k}$, $k = \overline{1,N/2-1}, d_{N/2} = 0, d_N = 0$ (see Fig. 2.)

In Fig. 1., Fig. 2. by $N = 80, L = 1$ there are represented discrete eigenvalues for $(-u'')$, imaginary parts of the discrete eigenvalues for u' by different values of $n = 1;2;3;4;30$ and corresponding, first $N = 80$ modified continuous values for the FDSES method.

For N- odd we obtain:

1) $d_k = \lambda_{2,k}, k = \overline{1,(N-1)/2}, d_{k+(N-1)/2} = \lambda_{2,(N-1)/2-k+1}, k = \overline{1,(N-1)/2}, d_N = 0,$

2) $d_k = \lambda_{1,k}, k = \overline{1,(N-1)/2}, d_{k+(N-1)/2} = -\lambda_{1,(N-1)/2-k+1}, k = \overline{1,(N-1)/2}, d_N = 0.$

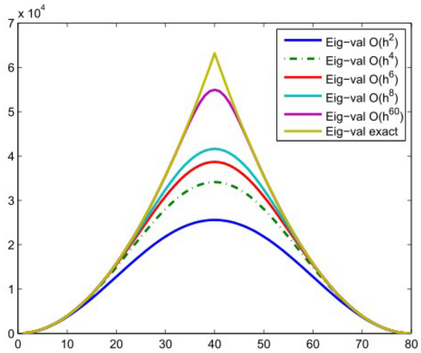

Fig. 1. Eigenvalues for $-u''(x)$ depending on x at $N = 80, L = 1, n = 1;2;3;4;30$.

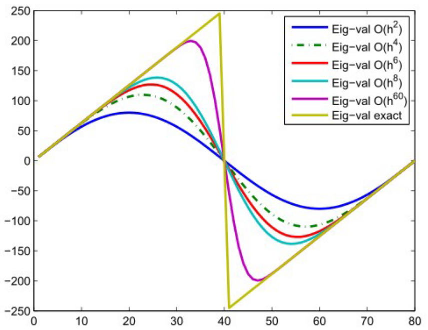

Fig. 2. Imaginary part of eigenvalues for $u'(x)$ depending on x at $N = 80, L = 1, n = 1;2;3;4;30$.

For variable coefficients k, p, q numerical solutions of the discrete heat transfer problems are obtained with MATLAB ODEs solver "ode15s", using the spectral representation of matrices $A_{m,n}, m = 1;2$ (methods FDS, FDSES and versions of differentiation matrices).

3. Results and discussion

The previously described methods of derivative approximation, like other finite difference approximation, can be applied in this chapter to estimate function's derivatives by solving ordinary and partial difference equations with variable coefficients, solving linear heat transfer equations with the homogenous boundary conditions of the first kind and solving nonlinear heat transfer equation with the boundary conditions of the first kind.

3.1 Ordinary differential equations (ODEs) with variable coefficients

The described finite differences can be used to solve numerically ODEs in the form:

$$\begin{cases} k \cdot u''(x) + p \cdot u'(x) + q \cdot u(x) = f(x), x \in (0,L), \\ u(0) = u(L), u'(0) = u'(L). \end{cases} \quad (2)$$

For functions $k = 1, p = p(x), q = q(x)$ the finite difference equation (the linear system of algebraic equations) is $A_{2,n}U + (P * A_{1,n})U + QU = F$, where U, F are the column-vectors of N order, P, Q are N - order diagonal matrices with corresponding elements $p_j = p(x_j), g_j = q(x_j)$.

The matrices $A_{2,n}, A_{1,n}$ we can form using the spectral decomposition $A_{2,n} = WD_{2,n}W^*, A_{1,n} = WD_{1,n}W^*$ in two way:

1) for the multi-point stencil FDS diagonal matrices $D_{2,n}, D_{1,n}$ with elements $\mu_{2,k}, \mu_{1,k}$,

2) for the FDSES diagonal matrices $D_{2,n}, D_{1,n}$ with elements $\lambda_{2,k}, \lambda_{1,k}, k = \overline{1,N}$ in special way.

We can find the vector U in the form $U = A^{-1}F$ or in MATLAB $U = A\backslash F$, where $A = A_{2,n} + P * A_{1,n} + Q$.

Special test-examples (Example 1, Example 2 and Example 3) were created with the aim of evaluating the accuracy of the FDS and FDSES methods.

Example 1.

The boundary value problem (2) was solved for constant coefficients: $p = 3, q = -1, f(x) = \cos(4\pi x) - 3\sin(28\pi x)$, $L = 1$. The exact solution is $u(x) = u_1(x) + 3u_2(x)$, where

$$u_1(x) = \frac{-(16\pi^2+1)\cos(4\pi x) + 12\pi\sin(4\pi x)}{(16\pi^2+1)^2 + (12\pi)^2},$$

$$u_2(x) = \frac{84\pi\cos(28\pi x) + (784\pi^2+1)\sin(28\pi x)}{(784\pi^2+1)^2 + (84\pi)^2}.$$

Several maximal errors of solutions were computed corresponding to different n values, namely $n = 1$ (the standard FDS in 3-point stencil), $n = 7, n = 15$ and FDSES. For $N = 35$ we have $5.3e-04 (n = 1), 4.57e-06 (n = 15), 1.5e-15$ (FDSES).

The spectral method for $N \geq 28$ is exact, because of the given linear combination for functions $\sin(p\pi x), \cos(p\pi x), p \leq 28$.

Example 2.

The boundary value problem (2) was solved for $p(x) = 4k_0\pi\cos(2k_0\pi x), q(x) = -(2k_0\pi)^2(\sin(2k_0\pi x) - \cos^2(2k_0\pi x)), f(x) = f_1(x)f_0(x), f_0(x) = \exp(-\sin(2k_0\pi x)),\ f_1(x) = \cos(4\pi x)$. We can see, that function $v(x) = u(x)/f_0(x)$ is the solution for

ODEs of constant coefficients $v''(x) = f_1(x), x \in (0,1)$ with periodic BCs.
This solution is in the following form: $v(x) = -\frac{\cos(4\pi x)}{16\pi^2} + C$, where C is an arbitrary constant.
The exact solution is $u(x) = -\frac{\cos(4\pi x)}{16\pi^2} f_0(x) + C f_0(x)$, where from $u(0) = 0$ follows that $C = \frac{1}{16\pi^2}$.
In the TABLE 1 there are represented the maximal errors of solutions by $N = 20,40,80,100,160; k_0 = 2,8,14$, obtained with global (FDSES= DM) and local approximations for different n (1;2;10).

TABLE 1 THE MAXIMAL ERRORS OBTAINED WITH GLOBAL (FDSES= DMS) AND LOCAL APPROXIMATIONS FOR DIFFERENT N

k_0		N	(FDSES)	n=1	n=2	n=10
2		20	4.5e -6	1.1e -2	2.1e -3	3.3e -5
2		40	2.2e -11	2.5e -3	1.6e -4	1.1e -8
8		80	2.6e -6	7.1e -3	1.5e -3	4.1e -5
8		100	4.7e -8	7.0e -2	9.4e -3	1.0e -5
14		140	3.0e -7	1.4e -2	3.0e -3	7.7e -7

The solutions by $N = 80, k_0 = 8, n = 1; 2$ are represented in Fig. 3., Fig. 4.

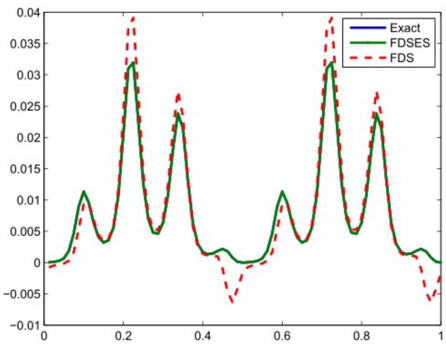

Fig. 3. u=u(x) − solution with the exact method, with FDS (n=1, error =7.11e−03) and FDSES (error =2.61e−06) methods, $k_0 = 8, N = 80$.

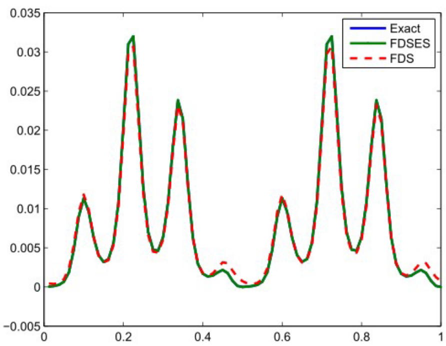

Fig. 4. u=u(x) − solution with the exact method, with FDS (n=2, error =1.30e−03) and FDSES (error =2.61e−06) methods, $k_0 = 8, N = 80$.

3.2 Solving linear partial differential equations (PDEs) with variable coefficients

For the heat transfer equation (1) using method of lines (MOL) we obtain the linear discreate system of ODEs:

$$\dot{U}(t) = (K * A_{2,n})U(t) + (P * A_{1,n})U(t) + QU(t) + F(t), U(0) = U_0,$$

where U, F, U_0 are N-order column-vectors with the elements $u_j(t) \approx u(x_j, t), f_j(t) = f(x_j, t), u_j(0) = u_0(x_j)$, K, P, Q are N-order diagonal matrices with the elements $k(x_j), p(x_j), q(x_j), j = \overline{1, N}$.

Formed the matrices with the spectral decomposition we obtain FDS and FDSES approximations for the linear system of ODEs. If the coefficients k, p, q are costants, then solution we can obtain analytically, using Fourier methods.
In the case of variable coefficients MATLAB solver "ode15s" was used.
Example 3. Solving an example 2 with $k(x) = 1, f(x,t) = -f_1(x) f_0(x), L = 1, u_0(x) = \frac{1}{16\pi^2} f_0(x), k_0 = 2, N = 20$ we obtain the stationary solution by $t = 0.1$ with 42 times step for FDSES (max. error $4.0\,e-06$), FDS $(n \geq 7)$ (max. error $2.0\,e-05$) and with 45 times step for FDS $(n = 1)$, (max. error $3.0\,e-03$).
In Fig. 5., Fig. 6. there are represented the stationary solutions by $k_0 = 14, n = 1; 2, N = 160$ obtaining with FDS and FDSES (t=0.1).
In Fig. 7., Fig. 8. there are represented solutions $u = u(x,t)$ by $k_0 = 8; 14, N = 80; 160$ obtaining with FDSES (t=0.04).

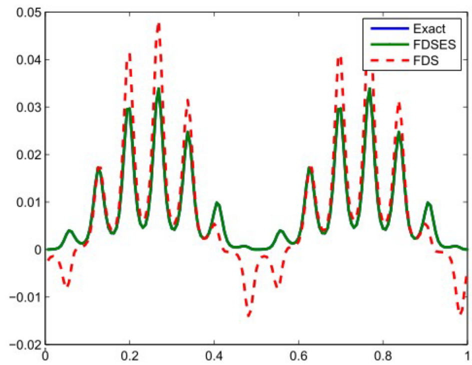

Fig. 5. u=u(x) − solution with the exact method, with FDS (n=1, error =1.44e−02) and FDSES (error =3.10e−07) methods, $k_0 = 14, N = 160$.

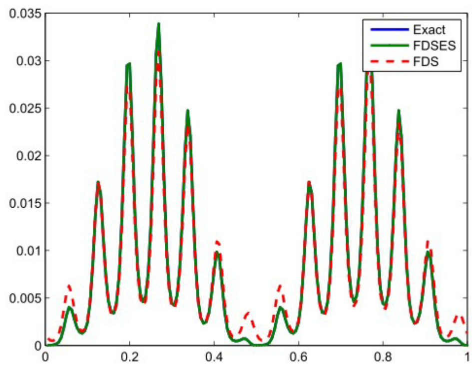

Fig. 6. u=u(x) − solution with the exact method, with FDS (n=2, error =2.95e−03) and FDSES (error =3.10e−07) methods, $k_0 = 14, N = 160$.

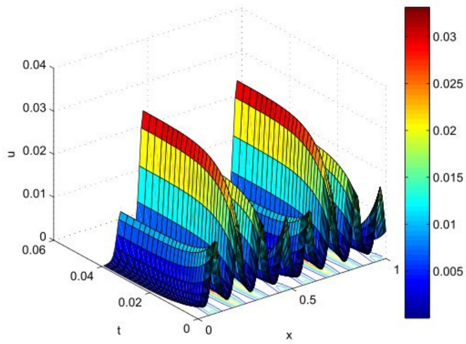

Fig. 7. u(x,t) – solutions with the FDSES method, $k_0 = 8, N = 80$.

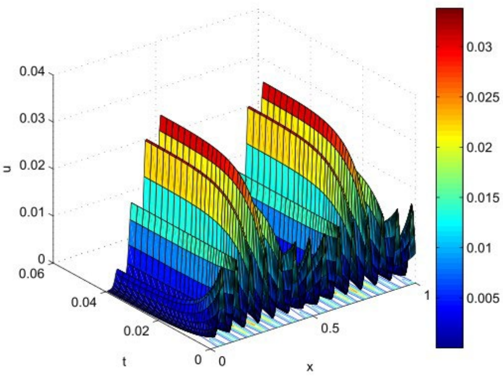

Fig. 8. u(x,t) – solutions with the FDSES method, $k_0 = 14, N = 160$.

The function $v(x,t) = u(x,t)/f_0(x)$ is solution from the heat transfer equation $v(x,t)_t = v(x,t)_{xx} - f_1(x)$ with periodic BCs. the exact solution by $v(x,0) = \frac{1}{16\pi^2}$ is $v(x,t) = f_1(x)(\exp(-16\pi^2 t) - 1)/(16\pi^2) + \frac{1}{16\pi^2} \to \frac{1-f_1(x)}{16\pi^2}$ if $t \to \infty$.

3.3 *Linear heat transfer equation with the homogenous boundary conditions (BC) of the first kind (FBCs)*

We consider the linear initial - boundary heat transfer problem in following form:

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\bar{k}\frac{\partial T(x,t)}{\partial x}\right) + f(x,t), \\ T(0,t) = 0, T(L,t) = 0, t \in \left(0, t_f\right), \\ \quad T(x,0) = T_0(x), x \in (0,L), \end{cases} \quad (3)$$

where $\bar{k} > 0$ is the constant parameter, t_f is the final time, T_0, f are given functions.
We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$. Using the finite differences of second order approximation we obtain from (3) the initial value problem for system of ordinary differential equations (ODEs) in the following matrix form

$$\begin{cases} \dot{U}(t) + \bar{k}AU(t) = F(t), \\ \quad U(0) = U_0, \end{cases} \quad (4)$$

where $A = A_{2,n}$ is the 3-diagonal matrix of $N-1$ order, $U(t), \dot{U}(t), U_0, F(t)$ are the column-vectors of $N-1$ order with elements $u_j(t) \approx T\left(x_j, t\right)$,
We can consider the analytical solutions of (3) using the spectral representation of matrix $A = WDW^T$.
The corresponding discrete spectral problem $Aw^k = \mu_k w^k, k = \overline{1, N-1}$ have following solution $\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{2N}$ (elements of the diagonal matrix D), $w_{i,j} = \sqrt{\frac{2}{N}}\sin\frac{\pi ij}{N}, i,j = \overline{1, N-1}$
(elements of the symmetric matrix W).
From transformation $V = W^T U (U = WV)$ follows the seperate system of ODEs

$$\begin{cases} \dot{V}(t) + \bar{k}DV(t) = G(t), \\ \quad V(0) = W^T U_0, \end{cases} \quad (5)$$

where $V(t), \dot{V}(t), V(0), G(t) = W^T F(t)$ are the column-vectors of $N-1$ order with elements $v_k(t), \dot{v_k}(t), v_k(0), g_k(t), k = \overline{1, N-1}$.
The solution of this system is the function

$$v_k(t) = v_k(0)\exp(-\kappa_k t) + \int_0^t exp\left(-\kappa_k(t - \tau)\right)g_k(\tau)d\tau, \quad (6)$$

where $\kappa_k = \bar{k}\mu_k$.
The analytical solution of heat transfer problem (3) by $\bar{k} = L = 1, f = 0, T_0 = 1$ with discontinuous initial and boundary data can obtained from following Fourier series:

$$T(x,t) = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{1}{2i+1}\exp(-(2i+1)^2\pi^2 t)\sin\left((2i+1)\pi x\right).$$

The corresponding solution with FDS (4) is in the following form:
$U(t) = Wexp(Dt)WU_0$,
where U_0 is the column vector with ones, the diagonal matrix D contain the discrete eigenvalues $\mu_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right), k = \overline{1, N-1}$.
For the FDSES the elements of matrix D are replaced with the first $N-1$ continuous eigenvalues $\lambda_k = \frac{k^2\pi^2}{L^2}$. The maximal error by $t = 0.02, N = 10$ is 0.089 for FDS and 0.0102 for FDSES. The results obtained with Fourier series contain on $x = 0, x = L$ oscillations (Gibbs phenomenon). For FDSES method these oscillations disappear. The maximal error by $t = 0.9, N = 10$ is 0.000012 for FDS and 0.000001 for FDSES.
From Fourier series $T(x,t) = \sum_{k=1}^{\infty} a_k(t)w_k(x), f(x,t) = \sum_{k=1}^{\infty} b_k(t)w_k(x), b_k(t) = (f, w_k)_*$
follows ODEs $\dot{a_k}(t) = -\bar{k}\lambda_k a_k(t) + b_k(t)$ with $a_k(0) = (T_0, w_k)_*, \lambda_k = \left(\frac{\pi k}{L}\right)^2$.
We have following solutions

$$a_k(t) = \exp\left(-\bar{k}\lambda_k t\right)a_k(0) + \int_0^t \exp\left(-\bar{k}\lambda_k(t-\xi)\right)b_k(\xi)d\xi.$$

From the discrete case (FDS) the solution of the matrix equation (4) is

$$U(t) = \exp(-\bar{k}tA)\,U(0) + \int_0^t \exp\left(-\bar{k}A(t-\xi)\right)F(\xi)d\xi.$$

Using the matrix A representation $A = WDW$ and transformation $V = WU$, follows (that for every matrix function $f(A) = Wf(D)W$)

$$V = \exp(-\bar{k}tD)\,V(0) + \int_0^t \exp\left(-\bar{k}D(t-\xi)\right)G(\xi)d\xi.$$

Therefore, we have the solution in the same integral form (6).

3.4 Nonlinear heat transfer equation with the boundary conditions (BC) of the first kind

We shall consider the dimensionless initial - boundary value problem (IBVP) of the first kind for solving the following nonlinear heat transfer equation in:

$$\frac{\partial T}{\partial t} = \frac{\partial^2(g(T))}{\partial x^2} + f(T), T(0,t) = 0, T(L,t) = 0, T(x,0) = T_0(x), \qquad (7)$$

where $T = T(x,t)$ is the solution of IBVP, $g(T)$ is nonlinear continuously differentiable power function with $\frac{\partial g}{\partial T} = g'(T) > 0$, $f(T)$ is nonlinear continuous source function.

For the power functions

$g(T) = T^{\sigma+1}, g'(T) = (\sigma+1)T^{\sigma}, f(T) = aT^{\beta}$, $a > 0, \beta \geq 1, \sigma \geq 0,$ $T_0(x) \geq 0$ follows $T(x,t) \geq 0$ for all $t \geq 0$.

From $T(0,t) = T(L,t) = 0$ follows that $T'(u) = 0$ by $x = 0$; $x = L$ and the solution of the problem (7) is not a classical.

In paper [12], [13] it is proved that

1) by $\beta < \sigma + 1$ exists a global bounded solution for all t,

2) by $\beta \geq \sigma + 1$ exists a global bounded solution for sufficient small $||T_0||$ ($||T_0||$ - the norm of the function $T_0(x)$, $||T_0||$=max $T_0(x)$ for $x \in (0,1)$), but for larger $||T_0||$ there exists the finite value T_* when $T(x,t) \to \infty$ if $t \to T_*$ ("blow up" solutions).

The IBVP (7) in the vector form is represented in the following way

$$\dot{U} + AG = F, U(0) = U_0, \qquad (8)$$

where $A = PDP$ is the standard 3-diagonal matrix of $N-1$ order with elements $\frac{1}{h^2}\{-1; 2; -1\}$, G, F are the vectors-column of $N-1$ order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t)), k = \overline{1, N-1}$,

$U(t), \dot{U}(t), U_0, F(t)$ are the column-vectors of $N-1$ order with elements $u_j(t) \approx T(x_j,t)$,

$\dot{u_j}(t) \approx \frac{\partial T(x_j,t)}{\partial t}$, $u_j(0) = U_0(x_j), f_j(t) = f(x_j,t), j = \overline{1, N-1}$.

The numerical experiment with $L = 1$ and $T_0(x) = x(1-x) \geq 0$ was produced by MATLAB solver "ode23s" for different values of σ and β [11].

For, example, by $a = 5, \sigma = \beta = 3, (\beta < \sigma + 1), t = 10, N = 6,10,20$ there were obtained following maximal errors with FDS and FDSES methods:

1) $N = 5 -$ 0,0125 (FDS), 0.0011$(FDSES)$;

2) $N = 10 - 0.0046(FDS), 0.0003(FDSES)$;

3) $N = 20 - 0.0013(FDS), 0.0001(FDSES)$.

In the Fig. 9., Fig. 10. we can see two type of solutions for three-time moments $(t = 0, t = T1, t = T2 > T1)$ by $\sigma = 3$ depending on the parameters β, a, obtained with the FDSES method $(N = 80)$:

1) $\beta = 4(\beta = \sigma + 1), a = 12$, three-timepoints of the numerical experiment - (0, 17.800, 17. 813000), the solution $u(x,t) \to \infty$ globally for all $x \in (0,1)$, when $t \to T_* < \infty$ ($T_* = 17.813001$ is finite value of time, this is the global "blow up" solution), (Fig. 9.).

2) $\beta = 5(\beta > \sigma + 1), a = 50$, three-time points of the numerical experiment -
(0, 16.018700, 16.018780), the solution $u(x,t) \to \infty$ locally neighbourhood of point $x = 0.5$, when $t \to T_* < \infty$ (for finite value of $T_* = 16.018781$, this is the local "blow up" solution), (Fig. 10.).

It should be noted that when performing numerical experiments directly at the time values
$T_* = 17.813001$ and $T_* = 16.018781$, the MATLAB solver "ode15s" stopped working.

The maximum value of the function $u(x,t)$ (max(u)=2060.6247) is shown in Fig. 11., Fig. 12., calculated at the parameter values $\beta = 5, \sigma = 3, a = 15$ at the time moment t =0.0278000. In this case, the local "blow up" solution constant is T_*=0.0278588.

The numerical experiments presented in the given publication were performed with the MATLAB solver "ode15s".

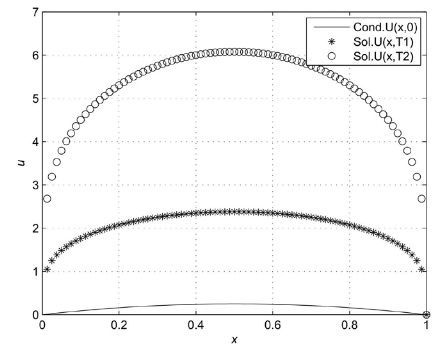

Fig. 9. u(x,t) - solutions for fixed t at (0, 17.800, 17. 813) and for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 12$.

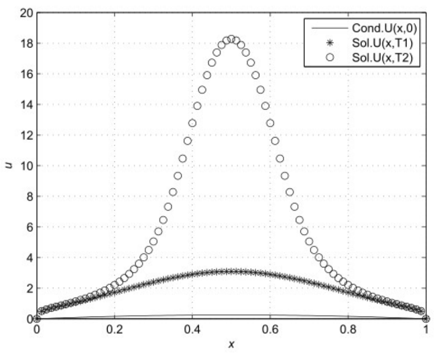

Fig. 10. $u(x,t)$ - solutions for fixed t at (0, 16.01870, 16.01878) and for $x = 0.5, \beta = 5, \sigma = 3, a = 50$.

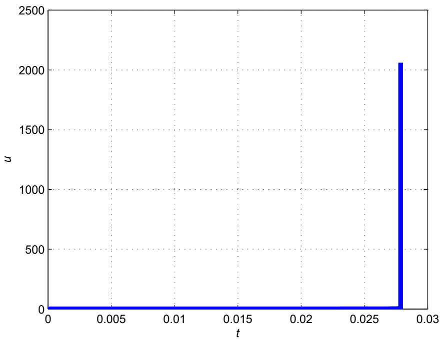

Fig. 11. $u(x,t)$ - solution depending on t at fixed x=0.5, its maximal value $max(u)$=2060.6247 at t =0.0278, $\beta = 5, \sigma = 3, a = 15$.

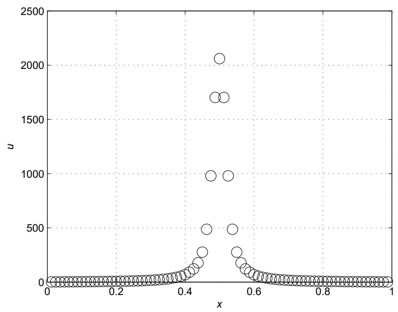

Fig. 12. $u(x,t)$ - solution depending on x at fixed t= 0.0278, its maximal value $max(u)$=2060.6247 at x=0.5, $\beta = 5, \sigma = 3, a = 15$.

In the monograph [13] there are collected all methods used in the study of “blow-up” of solutions to mathematical models of realistic physical phenomena. The “blow-up” effect occurs, for example, when a computer breaks down as a result of an electrical breakdown, or when a nuclear bomb explodes, and in several other interesting physical phenomena. Here exist several well-known methods of the study of the “blow-up” effect of solutions to nonlinear equations.

Conclusions

1. It was found that the FDSES method and the trigonometric interpolation method were equivalent methods in terms of accuracy.

2. All eigenvalues and eigenvectors for finite difference operators have been obtained. In the case of the homogenous first-kind boundary conditions, the FDSES method provided a solution with any desired accuracy.

3. The algorithm of the discrete Fourier method has been formed in different wise - using the special transformation by reducing the vector problem to scalar separated problem with the discrete eigenvalues, using the complex discrete Fourier series and the real discrete Fourier series for the respective vector components.

4. The advantages of the FDSES method for solving the problems of ODEs with periodical boundary conditions have been demonstrated in comparison with local FDS methods. In solving the partial differential equations for the linear heat conduction equation using the method of lines, exact algorithms in space were obtained. The FDSES method proved to be useful in solving boundary value problems both with periodic boundary conditions and also with homogenous boundary conditions of the first type.

5. Linear and nonlinear partial differential equations have been solved using the FDSES method due to the MATLAB solvers “ode15s”, and "ode23s".

6. By the theoretical positions, the solution of the nonlinear problem “blow-up” has been studied and obtained in connection with different characteristic-parameter values of the nonlinear heat conduction equation with the above-mentioned high-accuracy solving methods, especially, the FDSES method.

7. Numerical experiments showed that obtaining the solution of the "blow-up" phenomena and its graphic visualization is possible only with high-precision numerical methods - the values of time moments in the process of mathematical modelling had to be chosen with an accuracy of no less than 1.e -5 or 1.e - 6.

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