The Heat Transfer and Magnetohydrodynamics Problem with Heat Source in Half Infinite 1-D Domain

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Abstract. **In this paper we consider the temperature and laminar flow of an incompressible conducting fluid past a non-conducting half-space. For the space approximation the finite differences method-finite difference scheme (FDS) and finite difference scheme with exact spectrum (FDSES) for solving the heat transfer and laminar flow initial boundary-value problem are used. This procedure allows reducing the problem to initial value problem for ordinary differential equations and the solution to the problem can be obtained numerically and analytically. The equation of the temperature is un-depending on the velocity and this function we can obtain in analytical form use the integral transform methods- Fourier and Laplace transforms.**

Keywords: 1-D MHD problems, FDS and FDSES methods, Fourier and Laplace transforms.

I. Introduction

Nature of fluids, hydrodynamics, differential equations, dimensional analysis, viscous flows and mathematical theory of fluid motion, useful in applications to both hydrodynamics is described in [6], [5]. Effective finite difference and conservative averaging methods for solving problems of mathematical physics are described in [16].

The distribution of electromagnetic fields, forces and temperature induced by the system of the alternating electric current in the conducting cylinder has been calculated in [15].

The 3-D MHD problem is analysed numerically in [13] and for solving of MHD problem of viscous incompressible fluid the special monotonous difference schemes (FDS, FDSES and others) have been developed in [14].

Using the hydrodynamics, magnetohydrodynamics (MHD) and heat transfer aspect [4], [8], [7], [9], [3] we consider simple problem of the laminar flow of an incompressible conducting fluid past a non-conducting space y∈(-∞; +∞).

The fluid flows through this space and in contact with the plane xz-plane. A constant magnetic field of strength H_0 acts in the z-direction. The magnetics Reynolds number of the flow R_m is assumed to be small [8]. Under these conditions all the considered functions at a given point in the space depend only on its y-coordinate and time $t \in [0, t_f]$ (t_f is the final time) and $V = (u(y,t), 0,0)$ is the vector velocity of the fluid with one component in x-direction.

The solutions of some problems of partial differential equations (PDE) with PBCs are obtained, using the method of lines (MOL) to approach the PDEs in the time and for discretization them in the space, applying the finite difference scheme with central differences of a second order of the approximation (FDS) and the finite difference scheme with the exact spectrum (FDSES).

II. Materials and methods

In the present chapter the dimensional and dimensionless problems are considered. For this purpose, the integral transform methods, FDS and FDSES methods for solving the heat transfer problem and the

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problem of the laminar flow of an incompressible conducting fluid velocity is studied.

A. The Dimensional Problems

Being based on the above-mentioned assumptions, we have [5]:

1) the magnetic induction has one non-vanishing component $B_z = \mu H_0 = B_0 = const$

2) the Lorentz force $F = JxB$ has one component in x-direction is $F_x = -\sigma B_0^2 u$,

3) the compatibility relation is $\rho_\infty - \rho = \rho\beta_0(T - T_\infty)$,

4) the equation of motion

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta_0(T - T_\infty) - \frac{\sigma B_0^2}{\rho}u,$$

5) the energy or heat equation with the source term

$$\rho C_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2} + Q,$$

6) let us introduce the following non-dimensional variables:

$$y' = \frac{U}{\nu}y, t' = \frac{U^2}{\nu}t, u' = \frac{u}{U}, p_r = \frac{C_p\nu}{k\rho},$$

$$M = \frac{\nu\sigma B_0^2}{\rho U^2}, Q' = \frac{\nu^2 Q}{kT_0U^2}, T' = \frac{T-T_\infty}{T_0},$$

$$Gr = \frac{\nu\beta_0 g(T - T_\infty)}{U^3}.$$

Here $\rho, \mathrm{H}, \mathrm{B}, \mathrm{J}, \beta_0, C_p, k, \sigma, Q, \nu, g$ are fluid density, vectors of magnetic field intensity, magnetic induction, conduction electric density, coefficient of volume expansion, specific heat at constant pressure, thermal conductivity, electric conductivity, intensity of the applied heat source, kinematics viscosity, acceleration due to gravity, $T_\infty, T_w, T_0 = T_w - T_\infty$ are temperature of the fluid away from the plane surface, temperature of the plane surface, reference temperature, p_r, Gr, M are Prandtl, Grashof and Magnetic (Stewart) numbers. This is the well-known Boussinesq approximation.

We shall consider a point-type heat source of the form $Q = Q_0\delta(y)H(t)$, where $\delta(y), H(t)$ are δ- Dirac and Heaviside functions.

B. The Dimensionless Problem

The dimensionless problem for heat transfer and velocity equations is:

$$\begin{cases} p_r\frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 T(y,t)}{\partial y^2} + Q_0\delta(y)H(t), y \in (-\infty,\infty), t > 0, \\ \frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} + GrT(y,t) - Mu(y,t), \\ T(y,0) = u(y,0) = 0, u(\pm\infty,t) = T(\pm\infty,t) = 0. \end{cases} \quad (1)$$

The equation of the temperature is un-depending on the velocity and this function we can obtain in analytical form use the integral transformation methods. We can also use FDS method by $\pm\infty \approx \pm L$, where approximately $L = 5$.

C. The Integral Transform Method for Solving of Heat Transfer Problem

Using the integral Fourier transform [2] $T_*(k,t) = (2\pi)^{-0.5}\int_{-\infty}^{\infty} T(y,t)\exp(-iky)\,dy$

we obtain

$$p_r\frac{\partial T_*(k,t)}{\partial t} = -k^2T_*(k,t) + Q_0H(t)/\sqrt{2\pi}, k \in (-\infty,\infty), t > 0 \quad (2)$$

The solution by $T_*(y,0) = 0$ is

$$T_*(k,t) = \frac{Q_0}{\sqrt{2\pi}}\frac{1-\exp(-\alpha k^2)}{k^2}, \quad (3)$$

where $\alpha = \frac{t}{p_r}$.

This solution we can obtain also with Laplace transform

$$T^*(y,s) = \int_0^\infty T(y,t)\exp(-st)\,dt.$$

Then from equation (1) follows

$$p_r sT^*(y,s) = \frac{\partial^2 T^*(y,s)}{\partial y^2} + Q_0\delta(y)/s$$

and from Fourier transform we get

$$p_r sT_*^*(k,s) = -k^2T_*^*(k,s) + \frac{Q_0}{\sqrt{2\pi}s}$$

or

$$T_*^*(k,s) = \frac{Q_0}{\sqrt{2\pi}s(p_r s + k^2)}.$$

Using the inverse Laplace transform we obtain (3).

With the inverse Fourier transform

$$T(y,t) = (2\pi)^{-0.5}\frac{Q_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1-\exp(-\alpha k^2)}{k^2}\exp(iky)\,dk$$

we get

$$T(y,t) = \frac{Q_0}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}F(\alpha), \text{ where}$$

$$F(\alpha) = \int_0^\infty \frac{1-\exp(-\alpha k^2)}{k^2}\cos(ky)\,dk, F(0) = 0.$$

Using derivation with respect to parameter α we obtain the known integral

$F'(\alpha) = \int_0^\infty \exp(-\alpha k^2)\cos(ky)\,dk = \sqrt{\frac{\pi}{4\alpha}}\exp\left(-y^2/(4\alpha)\right)$ or

$F(\alpha) = \sqrt{\pi/4}\int_0^\alpha \frac{\exp\left(-y^2/(4\xi)\right)}{\sqrt{\xi}}d\xi.$

With transformation $s = \frac{y^2}{4\xi}$ follows:

$$F(\alpha) = \sqrt{\frac{\pi}{16}}|y|\int_{y^2/(4\alpha)}^\infty \frac{\exp(-s)}{s^{1.5}}ds.$$

Using the integration by parts

$$\int v\,du = uv - \int u\,dv, du = s^{-1.5}ds,$$
$$v = exp(-s),$$

$dv = -exp(-s)ds, u = -2s^{-0.5}$, we obtain

$$F(\alpha) = \sqrt{\frac{\pi}{16}}|y|\left(4\sqrt{\alpha}\exp\left(-\frac{y^2}{4\alpha}\right)/|y| - 2\int_{y^2/(4\alpha)}^\infty \frac{\exp(-s)}{s^{0.5}}ds\right).$$

Therefore, we have obtained the analytical solution in the form:

$$T(y,t) = Q_0\sqrt{2/\pi}\left(\sqrt{\frac{t}{2p_r}}\exp\left(-\frac{y^2 p_r}{4t}\right) - |y|\sqrt{\frac{\pi}{8}}erfc\left(|y|\sqrt{\frac{p_r}{4t}}\right)\right),$$

$erfc(u) = \frac{2}{\sqrt{\pi}}\int_u^\infty \exp(-\xi^2)\,d\xi.$

D. The FDS and FDSES Methods and the Solution of the Heat Transfer Problem

We can construct the FDSES when in the representation for FDS, $A = WDW$ the diagonal elements of matrix D are replaced with the eigenvalues from the differential problem [11], [12].

For obtaining the temperature and velocity we consider uniform grid in the space $y_j = jh - L, j = \overline{0,2N}, Nh = L$.

Using the finite differences of second order approximation for partial derivatives of second order respect to y we obtain from the first equation of (1) the initial value problem for system of ODEs in the following matrix form

$$\dot{V}(t) + \frac{1}{p_r}AV(t) = \frac{Q_0}{p_r h}, V(0) = 0 \qquad (4)$$

where A is the 3-diagonal matrix of $2N-1$ order in the form

$A = \frac{1}{h^2}\cdot[-1;\ 2;\ -1\,]$,

$V(t), \dot{V}(t)$ are the column-vectors of $2N-1$ order with elements $v_j(t) \approx T(y_j,t))$, $\dot{v}_j(t) \approx \frac{\partial T(y_j,t)}{\partial t}$, $j = \overline{1,2N-1}$.

The expression of the vector Av can be represented in following way

$$Av_j = -\left(v_{j+1} - 2v_j + v_{j-1}\right)/h^2, j = \overline{1,2N-1}, \quad (5)$$

where v is the column-vector of $2N-1$ order with elements

$v_j, j = \overline{1,2N-1}, v_0 = v_{2N} = 0.$

Using two vectors v^1, v^2 scalar product

$$[v^1, v^2] = h\left(\sum_{j=1}^{2N-1} v_j^1 v_j^2\right)$$

it is possible to prove, that the operator A is symmetrical and $[Ay, y] \geq 0$ [1].

The corresponding discrete spectral problem $Aw^k = \mu_k w^k, k = \overline{1,2N-1}$ have following solution $\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{4N}$ (elements of the matrix D), $w_{i,j} = \sqrt{\frac{2}{L}}\sin\frac{\pi i j}{2N}$, $i,j = \overline{1,2N-1}$ (elements of the symmetrical matrix W).

Using the usual scalar product of two vectors for eigenvectors without the step h,

$(w^k, w^m) = \sum_{j=1}^{2N-1} w_j^k w_j^m = \delta_{k,m}$, $w_j^k = C_k\sin\left(k\pi(jh+L)/(2L)\right)$, we get $C_k = \sqrt{\frac{2h}{2L}} = \sqrt{\frac{1}{N}}$.

The solution of discrete boundary value problem $Av = f, v(-L) = v(L) = 0$, or of the finite difference scheme (FDS) with second order of approximation for the boundary value problem of differential equation (1D Poisson equation) $-u''(y) = f(y), u(-L) = u(L) = 0$,

we can write in following form $Av = WDW\,v = F$, where F is the column-vector of $f(y_j), j = \overline{1,2N-1}$.

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we can write in following form $Av = WDW\,v = F$, where F is the column-vector of $f(y_j), j = \overline{1,2N-1}$.

The solution of the spectral problem for the corresponding differential problem $-w''(y) = \lambda w(y), w(0) = w(2L) = 0$ is in following form:

$w^k(y) = \sqrt{\frac{2}{L}}\sin\frac{k\pi y}{2L}, \lambda_k = \left(\frac{k\pi}{2L}\right)^2, (w^k, w^m)_* = \int_0^L w^k(y)w^m(y)dy = \delta_{k,m}.$

We can construct the FDSES when in the representation $A = WDW$ the diagonal elements of matrix D are replaced with the eigenvalues λ_k from the differential problem. Then the matrix A is not in the 3-

diagonal form but this is full matrix and $WW = E, W^{-1} = W, A = WDW$, where $D = diag(\lambda_k)$.

The solution of the equation $WDWv = F$ we can obtain in the form $v = WD^{-1}WF$ or $v = A^{-1}$.

E. The FDS and FDSES Methods for Solving the Velocity of a Laminar Flow of an Incompressible Fluid

Using the finite differences of second order approximation (FDS) for partial derivatives of second order respect to y we obtain from the second equation of (1) the initial value problem for system of ODEs in the following matrix form

$$\dot{U}(t) + AU(t) + MU(t) = GrV(t), U(0) = 0, \quad (6)$$

where A is the 3-diagonal matrix of $N-1$ order, $U(t), \dot{U}(t), V(t)$ are the column-vectors of $2N-1$ order with elements

$$u_j(t) \approx u(y_j(t)), \dot{u}_j(t) \approx \frac{\partial u(y_j,t)}{\partial t}, j = \overline{1, N-1}, v_j(t) \approx T(y_j(t)), j = \overline{1, 2N-1}.$$

The solution with FDSES method is obtained in the representation $A = WDW$ where the diagonal elements of matrix D are replaced with the eigenvalues λ_k from the differential problem.

Systems of ODEs (4), (6) are solved with Matlab routine "ode15s".

III. Results and Discussion

In the present chapter we have solved the 1-D boundary value problem for Poisson equation, and have calculated the temperature and velocity of the non-compressible liquid flowing under the influence of the magnetic field.

The solution of 1-D Poisson equation (chapter D) for function $f(y) = \pi^2 sin(\pi(y+L)/2)$ is in the form $v(y) = -4sin(\pi(y+L)/2)$.

We have for $y \in [-L, L], y(-L) = y(L) = 0, N = 20$ following maximal errors $E_r: E_rFDS = 0.0518$ for FDS and $E_rFDSES = 10^{-14}$ for FDSES see (Fig. 1., Fig. 2.). In the Fig. 2. the error E_rFDS is compared with the error E_rMat, obtaineed by matrix A solutions Av=F in the form $v = A^{-1}F$.

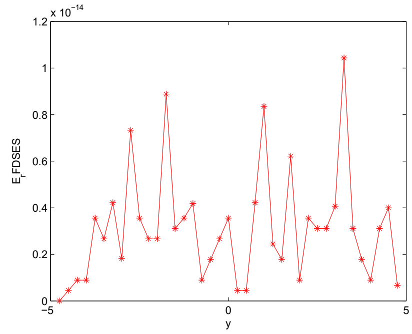

Fig. 1. The error for FDSES by N=20, L=5.

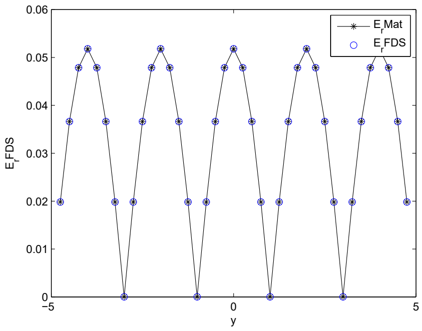

Fig. 2. The error for FDS and for matrix solution by N=20, L=5.

The solutions for temperature and velocity for $N = 40, Q_0 = 1, Gr = 10, M = 1, p_r = 0.71, t_f = 2$ using Matlab are obtained with following results:

a) the maximal values for temperature by FDS method $max(T_{apr})$ and by analytical (exact) method $max(T_{ex})$ are: $max(T_{apr}) = 0.9456$, $max(T_{ex}) = 0.9469$;
b) the maximal values for velocity using FDS method $max(u_{FDS})$ and by using FDSES method $max(u_{FDSES})$ are: $max(u_{FDS}) = 0.0310, max(u_{FDSES}) = 0.0311$.

The solutions of the temperature T(y,t) , $t \in [0, t_f]$ and $t = t_f$ are represented in Fig. 3.-Fig. 6., the corresponding solutions of velocity u(y,t) are represented in Fig. 7.-Fig. 10.

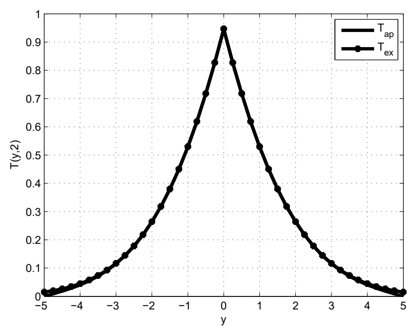

Fig. 3. The solutions $T(y, 2)$ depending on y.

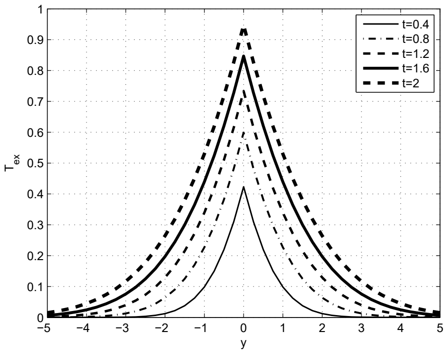

Fig. 4. The exact solutions $T(y,t)$ depending on t.

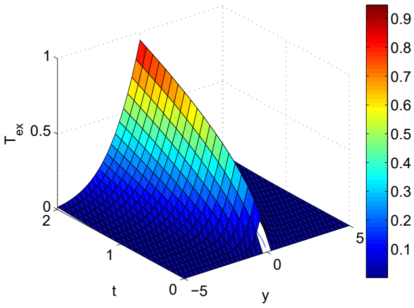

Fig. 5. The exact solutions $T(y,t)$ depending on y and t.

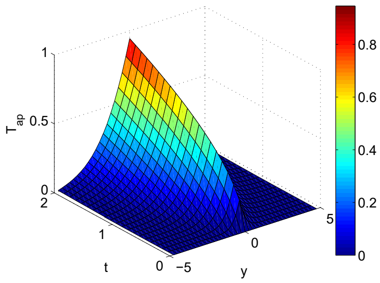

Fig. 6. The approximate solutions $T(y,t)$ depending on y and t.

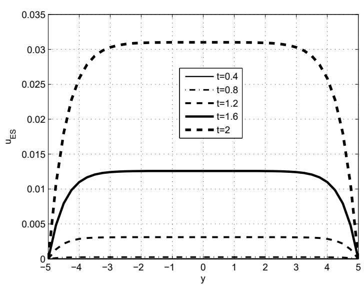

Fig. 7. The FDSES solutions $u(y,t)$ depending on t.

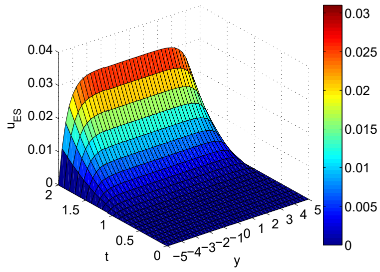

Fig. 8. The FDSES solutions $u(y,t)$ depending on y and t.

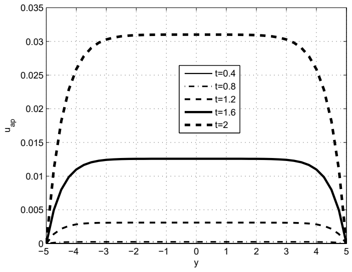

Fig. 9. The FDS solutions $u(y,t)$ depending on t.

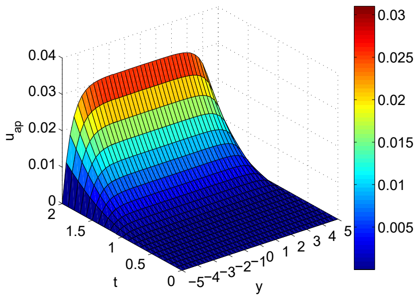

Fig. 10. The FDS solutions $u(y,t)$ depending on y and t.

We can see that the maximal values of temperature and velocity are concentrated around the point y=0. In the Fig. 11 the table (Tab) of maximum values u(y,t) depending on Gr and M for the values (1,2. . .10), $Q_0 = 1$, is presented. It is apparent that velocity decreases if M increases and velocity is growing up if Gr and temperature increases. In turn, the temperature is growing up with increasing Q_0, for example, if $Q_0 = 10$, then $max(T_{ex}) = 9.470$

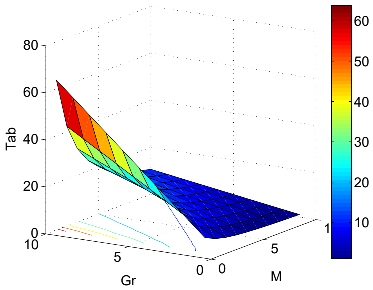

Fig. 11. The maximal values (Tab) of FDSES solutions for $u(y,t)$ depending on M, Gr.

IV. Conclusions

1. The approximation of corresponding initial boundary value problem of the system of PDEs is based on the finite difference schemes FDS and FDSES.

2. The solutions of temperature and velocity have been obtained depending on time and space parameters.

3. The max absolute value of difference between corresponding numerical and analytical solutions of velocity was approximately 0.1%.

4. The velocity and temperature have symmetrical profile depending on y, the maximum of temperature and velocity is concentred around the point y=0.

5. The solutions obtained by the MHD problems studied illustrate the simplicity and flexibility of the finite difference schemes FDS and FDSES in terms of their applicability and accuracy.

6. The system of parabolic type equations has been solved depending on time using Matlab routine "ode15s".

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