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On the Ginzburg-Feinberg Problem of Frequency Electromagnetic Sounding for Unambiguous Determination of the Electron Density in the Ionosphere

Sharif E. Guseynov[1, 2, 3,a], Janis S. Rimshans[1,2]
[1]*Institute of Fundamental Science and Innovative Technologies, Liepaja University, Liepaja LV-3401, Latvia;*
[2]*Faculty of Science and Engineering, Liepaja University, 4 Kr.Valdemar Street, Liepaja LV-3401, Latvia;*
[3]*"Entelgine" Research & Advisory Co., Ltd., Kleistu Street 2-53, Riga LV-1067, Latvia; [a]sh.e.guseinov@inbox.lv*

Abstract. In the present work, we investigate an inverse problem of frequency electromagnetic sounding for unambiguous determination of the electron density in the ionosphere. Direct statement of this problem is known as the Ginzburg-Feinberg problem that has, in general case, an essential nonlinearity. Inverse statement of the Ginzburg-Feinberg problem has the boundary-value formulation relative to two functions: the sought-for electric-field strength and the distribution of the electron density (or rather two-argument function appearing in the additive decomposition formula for distribution of the electron density) in the ionosphere. In the present work, we prove the existence and uniqueness of the solution of the Ginzburg-Feinberg problem as well as we propose the analytical method, permitting: first, to reduce it to the problem of integral geometry, and, thereupon, having applied the adjusted variant of the Lavrentiev's theorem, to reduce the obtained problem of integral geometry to the first kind matrix integral equation of Volterra type with a weak singularity.

Keywords: frequency electromagnetic sounding, electron density, Ginzburg-Feinberg problem, inverse boundary-value problem.

I. INTRODUCTION

The term "ionosphere" was introduced by Scottish physicist Sir R.A. Watson-Watt in 1926 in one of his letters, which was published only in 1969 in the English multidisciplinary scientific journal "Nature Magazine" ([1]). Apparently, the beginning of the history of studies of ionosphere is tied to the work of Italian electrical engineer and radio technician G. Marconi (awarded Nobel prize in Physics in 1909), who, in the end of December 1901, conducted a unique experiment and managed to get transatlantic radio signal, using a 152m tall antenna. In 1902, English physicist O. Heaviside supposed that the atmosphere has an ionized layer. His theory claimed that a radio signal can travel across the Earth despite its curvature. Independently from O. Heaviside, American electrical engineer A,E, Kennelly was conducting his own experiments to study transmission of short waves over Atlantics ([2]). The experiments conducted by O. Heaviside and A.E. Kennelly indicated that somewhere around the Earth there should an ionized layer of the atmosphere, which is capable of reflecting radio waves (now that layer is named "Kennelly-Heaviside layer" or just "E-region"). Possibly, the ideas of O. Heaviside and A.E. Kennelly together with the law of radiation of black body, which was formulated by German theoretical physicist M. Planck (awarded Nobel prize in Physics in 1918), contributed to the rapid development of radio astronomy starting with 1932. Also, it served as the founding point in the development of high frequency systems of the receiver-transmitter. In 1924, English physicist E.V. Appleton (awarded Nobel prize in Physics in 1947) together with his colleagues from the Cavendish Laboratory, University of Cambridge, and King's College London conducted the famous experiment (having strongly defined the theoretical basis for it, presenting that in advance) when the approximate height of ionosphere was detected for the first time ever. Then, in 1924-1927, Sir E.V. Appleton did prove the existence of the Earth's ionosphere, focusing his work and studies on that fundamental issue. In 1924, American geophysicist and engineer (one of the most notable specialists in the domains of Earth magnetism and ionosphere) L.V. Berkner did measure electronic density of ionosphere for the first time. Later, in 1933-1934, together with the colleagues from the Brookhaven National Library, Long Island, USA ([3]-[6]) he studied the fundamental physical issues tied to the effect of magnetoionic birefringence effect of radio waves in ionosphere and magnetoionic splitting of E-region, using critical frequency. Also, they proved the existence of substratification of F-region (proved the existence of the F_1 layer under the

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main F_2 layer). The results obtained by L.V. Berkner later appeared to be crucially important for creation of the short radio waves theory. Innovative studies of influential British radio physicist J,A, Ratcliffe and British physicist M.V. Wilkes (further – notable scientist in the domain of computer sciences) did stimulate creation of the theory of spread of very long radio waves in ionosphere ([7]-[10]). In 1940-1960, Soviet theoretical physicist and astrophysicist V.L. Ginzburg (awarded Nobel prize in Physics in 2003) developed the theory of spread of electromagnetic waves in plasma, in particular – in ionosphere.

In the current paper, authors study a non-linear problem of detection of density of electrons in ionosphere by the method of frequency electromagnetic probing. Its mathematical formulation can be found in the fundamental [11] and [12]. Apparently, authors of these monographs independently each from another did formulate and study the current non-linear problem, full solution of what is absent, as for now. In our paper, we study the issues of the existence and uniqueness of solution of the stated problem.

II. STRUCTURE AND KEY PARAMETERS OF THE IONOSPHERE, BASIC INVESTIGATION METHODS

Ionosphere is defined as border part of the Earth's atmosphere where the level of ionization is big enough to make notable impact on the spread of radio waves. ([4], [7], [11], [13], [14]). The lower border of ionosphere is at the height of around 50-60 km from the surface of the Earth, while its upper border is at the level of approximately 1000 km and transcends into plasmasphere and other magnetospheric plasma formations. The main parameters of ionosphere are its ionic composition, temperature, and concentration of electrons; and these parameters have complex dependence from the height. ([11], [13], [14]). If we are limited with consideration of only one parameter – concentration of electrons in ionosphere, then, usually, there are three spaces of the maximal concentration of electrons:

- D-region, which is about $60 \div 90$ km above the ground and has maximal concentration of electrons 10^3 cm^{-3};
- E-region, which is sometimes called Kennelly-Heaviside Layer, being $90 \div 120$ km above the ground and having maximal concentration of electrons 10^5 cm^{-3}. Such a dense concentration is achieved only in a thin space (as thin as $0.5 \div 1$ km), it is called E_s layer ($100 \div 110$ km above the ground);
- F-region, often called Appleton-Barnett layer, $120 \div 300$ km above the ground, maximal concentration of electrons 10^6 cm^{-3}. Such concentration of electrons is achieved in the so-called F_2 layer ($170 \div 300$ km above the ground), while in the layer F_1 ($120 \div 170$ km above the ground) the maximal concentration of electrons can be observed only in daytime, as it is caused by strong sun ultraviolet radiation, and it can drop up to 10^4 cm^{-3} level.

Worth underlining that the values mentioned above are presented as landmarks, since the heights of layers as well as concentration of electrons in them do experience strong regular sporadical ([11], [13]-[15]). Sporadical variations of the three main spaces of ionosphere – ion composition, temperature, and concentration of electrons – are tied to the interaction of particles and radiations generated in the sun or magnetospheric flare events. Sudden ionospheric disturbances in E- and D-regions are caused by X-ray burst generated in the Sun during chromospheric flares, which last only few minutes, while concentration of electrons in D-region can increase in dozen times; in E-region – it can increase twice. The effects and collateral phenomena are observed only in the enlightened part of ionosphere. When solar space beams reach the Earth, it causes an ionospheric disturbance known as Polar Cap Absorption (PCA). PCA belongs to D-region of ionosphere where concentration of electrons can double. Length of PCA is defined by the length of the event, which causes it. So, it can last up to several days. Development of auroral substorm causes notable changes in the entire ionosphere and changes the conditions of transmission of radio signals up to total absorption.

Now, the main methods of study if ionosphere are being presented. First, it is worth being mentioned that study of ionosphere is one of the actual scientific problems, which is tied both to the issues of fundamental problems of the physics of space plasma as well as to the applied problems. Impermanence of ionosphere (especially that in the high latitudes) attracted much attention, because of the importance of stable radio connection for both military and civil operations. Studies of ionosphere until the time, when opportunities for direct with rockets and satellites were provided, were based on its ability to absorb, reflect, and spread radio signals. The main methods of study of ionosphere are: method of particular electromagnetic probing; methods of vertical, incline, and reverse-incline probing; riometric and radiolocational methods; method of incoherent dissolution; method of spread of superlong waves. Worth mentioning that ionospheric methods are applied not only in the studies of the ionosphere itself, and its parameters, but also for studies of magnetospheric processes.

III. MATHEMATICAL STATEMENT OF THE ORIGINAL PROBLEM

As it was already mentioned in the introduction, concentration of electrons is one of the three main parameters for studying ionosphere ([7], [11], [16]). In the current section, we study the reverse problem for the sought-for concentration of electrons in the ionosphere with frequency electromagnetic probing of the ionosphere.

So, we approach the following boundary-value inverse problem: it is required to determine the function $E(x,z;\omega)$ and $V(x,z)$, where $(x,z;\omega)\in[0,L]\times\mathbb{R}^1_{++}\times[0,\omega_{\max}]$, which fit the equation

$$\left\|\nabla_{x,z}E(x,z;\omega)\right\|^2_{\mathbb{R}^2}-\varepsilon(x,z;\omega)=0,\quad x\in(0,L),\ z\in\mathbb{R}^1_{++},\ \omega\in(0,\omega_{\max}),\tag{1}$$

the Dirichlet boundary conditions

$$E(x,z;\omega)\big|_{(x,z)=(0,0)}=0,\ \omega\in[0,\omega_{\max}],\tag{2}$$

$$E(x,z;\omega)\big|_{(x,z)=(L_j,0)}=E_j(\omega),\omega\in(0,\omega_j)\ j=\overline{0,n},\tag{3}$$

and additional condition

$$0\le|V(x,z)|\ll U(z),\ x\in[0,L],\ z\in\mathbb{R}^1_{+},\tag{4}$$

where sought-for $E(x,z;\omega)$ is the electric-field strength; the dielectric permeability of the ionosphere is designated as

$$\varepsilon(x,z;\omega)\overset{def}{\equiv}\frac{m\cdot\left(\omega^2+\nu^2_{effective}\right)-4\cdot\pi\cdot e^2\cdot N(x,z)}{m\cdot\left(\omega^2+\nu^2_{effective}\right)};$$

m is the electron mass; $\omega=\omega(z)$ is the cyclic/angular frequency of the electromagnetic field; e is the elementary electronic charge; $\nu_{effective}$ is the effective number of collision of an electron with molecules or ions per second, and at each collision an electron on the average passes to a molecule or ion the pulse of the order $m\cdot\vec{r}^{\,2}$, where $\vec{r}$ means an ordered velocity imparted to the electron by electromagnetic field $\vec{E}$; $N(x,z)\overset{def}{\equiv}U(z)+V(x,z)$ is sought-for distribution of the electron density in the ionosphere, where $U(z)\in D^1\left(\mathbb{R}^1_{++}\right)$ is a priori given function, and $V(x,z)\overset{def}{\equiv}\sum_{i=0}^{n}\varphi_i(z)\cdot x^i$ is unknown; the boundary functions $E_j(\omega),\ \omega\in(0,\omega_j)\ \forall j=\overline{0,n}$ are experimentally measurable functions, and the essence of these values consists in the field phase, i.e. an eikonal (for instance, see [17]) at the measuring points $A_i(x=L_i,z=0),\ \forall i=\overline{0,n}$ relative to the phase of some measuring device located at the point $O(x=0,z=0)$; $\omega_{\max}\in\mathbb{R}^1_{++}$; $L\in\mathbb{R}^1_{++}$; $n\in\mathbb{N}$; $L_j\in(0,L)\ \forall j=\overline{0,n}$; $\omega_j\in(0,\omega_{\max})\ \forall j=\overline{0,n}$; $\mathbb{R}^1_{+}\overset{def}{\equiv}[0,+\infty)$; $\mathbb{R}^1_{++}\overset{def}{\equiv}(0,+\infty)$.

IV. TRANSFORMATION OF THE ORIGINAL PROBLEM TO A PROBLEM OF INTEGRAL GEOMETRY

First, shall we note that the additional information (4) cause study of the two mutually exclusive cases:

$$V(x,z)\equiv 0,\ (x,z)\in[0,L]\times\mathbb{R}^1_{++},\tag{5}$$

$$V(x,z)\ne 0,\ (x,z)\in[0,L]\times\mathbb{R}^1_{++}.\tag{6}$$

In the beginning, let us assume that the case (5) is the actual situation. Then, evidently, the initial problem (1)-(4) becomes direct: we need to define only the function $E(x,z;\omega)$. By the actual check, it is possible to find out that $\lambda(x)$ of the equation (1) is the solution for the Bernoulli equation ([18])

$$\lambda''(x)+C(z)\cdot\left(\lambda'(x)\right)^2=-A(z),\tag{7}$$

where

$$C(z)\overset{def}{\equiv}\frac{2\cdot\pi\cdot e^2\cdot U'(z)}{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)-4\cdot\pi\cdot e^2\cdot U(z)},$$

$$U(z)=\begin{cases}0,\ z\in[0,Z);\\ \left.\begin{array}{l}U(z)\in D^3(Z,+\infty),\\ 0<c\equiv const<U'(z),\end{array}\right\}\ z\in[Z,+\infty)\end{cases}\tag{8}$$

So, in the case (5) the problem of definition of the function $E(x,z;\omega)$ from the direct (1)-(4) results ([19]) to the problem of finding the function $\lambda(x)$ from (7), (8) with the boundary conditions

$$\lambda(x)\big|_{x=0}=0;\ \lambda(L)\big|_{x=0}=0.\tag{9}$$

Since the solution of the boundary-value problem (7)-(9) depends from the parameter ω, as the result we have set of characteristics $\lambda=\lambda(x;\omega)$, linking the point $O(x=0,z=0)$ of the location of the measurement device with the point $A(x=L,z=0)$ of the measurement, which depends on the cyclic frequency ω of the electromagnetic field $\vec{E}(x,z;\omega)$. Since the problem (7)-(9) is solved in quadrants, study of the case (5) is fully accomplished, and now we can switch to the study of the case (6). For that purpose, we introduce the functional first

$$\Omega[S(z)]\overset{def}{\equiv}\frac{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)-4\cdot\pi\cdot e^2\cdot S(z)}{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)}$$

and consider the equation

$$\left\|\nabla_{x,z} E_U\left(x,z;\omega\right)\right\|^2_{\mathbb{R}^2} = \Omega\left[U\left(z\right)\right]. \qquad (10)$$

Evidently that when $z_0 \in \left[0, Z\right)$ we can write

$$\frac{4\cdot\pi\cdot e^2\cdot V\left(x,z\left(x,z_0\right)\right)}{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U\left(z\left(x,z_0\right)\right)\right]} = O\left(\frac{V\left(x,z\left(x,z_0\right)\right)}{U\left(z\left(x,z_0\right)\right)}\right),$$

so consequently, with an accuracy up to infinitesimal of the second order of the fraction $\left[\frac{V\left(x,z\left(x,z_0\right)\right)}{U\left(z\left(x,z_0\right)\right)}\right]^2$ on the characteristic curves of the equation (10) there is an equality

$$\varepsilon\left(x,z;\omega\right) == \sqrt{\Omega\left[U\left(z\left(x,z_0\right)\right)\right]} - \frac{2\cdot\pi\cdot e^2\cdot V\left(x,z\left(x,z_0\right)\right)}{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)\cdot\sqrt{\Omega\left[U\left(z\left(x,z_0\right)\right)\right]}}.$$

Theorem (Lavrentyev, [20]). Let $u_i\left(x_1,x_2\right) = u_{i,1}\left(x_1,x_2\right) + u_{i,2}\left(x_1,x_2\right)$, $\left(i=1;2\right)$ where $u_{1,j}\left(x_1,x_2\right) \in D^2\left\{X_1\times X_2\right\}$, $\left(j=1;2\right)$ and function $u_{2,2}\left(x_1,x_2\right)$ are the solution of the equation

$$\left(\frac{u_{2,1}\left(x_1,x_2\right)}{\partial x_1}\right)^2 + \left(\frac{u_{2,1}\left(x_1,x_2\right)}{\partial x_2}\right)^2 = u^2_{1,1}\left(x_1,x_2\right). \quad (11)$$

Then, with an accuracy up to infinitesimal of the 2nd order of $u^2_{1,1}\left(x_1,x_2\right)$, we can present the function $u_{2,1}\left(x_1,x_2\right)$ as

$$u_{2,2}\left(A\right) = \int_{\Gamma_{OA}} u_{1,1}\left(x_1,x_2\right) d\gamma_{OA},$$

where Γ_{OA} as characteristic curve of the equation (11), linking points $O\left(x_1=0,x_2=0\right)$ and $A\left(x_1,x_2\right)$.

For being able to use the above mentioned Lavrentiev's Theorem in our study, we consider the solution of equation (1) in relation to the parameter ω at the point $\left(x=L, z=0\right)$, in advance switching the parameter ω for z_0: such substitution is possible, since from the evident formula

$$\sqrt{\frac{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U\left(z_0\right)\right]}{4\cdot\pi\cdot e^2}}\cdot\int_0^{z_0}\frac{d\xi}{\sqrt{U\left(z_0\right)-U\left(\xi\right)}} = \frac{L}{2},$$

which is the consequence of the equation (7) and from the fact that $\lambda\left(x\right)$ is symmetrical function in relation to the straight line $x=\frac{L}{2}$, which is

$$\lambda\left(\frac{L}{2}-x\right) = \lambda\left(\frac{L}{2}+x\right) \Leftrightarrow \lambda\left(x\right) = \lambda\left(L-x\right), \qquad (12)$$

so it is that each z_0 has only one value of $\omega = \omega\left(z_0\right)$ corresponding. Then, we can write that $E = E\left(L,0;z_0\right)$. An analogical approach lets us define the function $E_U = E_U\left(L,0;z_0\right)$ as the relevant solution of the equation (9) in relation to the parameter ω at the point $\left(x=L, z=0\right)$, also introducing in advance the parameter ω instead of z_0. Further, we introduce the function $E\left(L,0;z_0\right) \overset{def}{\equiv} E_V\left(L,0;z_0\right) + E_U\left(L,0;z_0\right)$, and again consider the Lavrentiev's Theorem. Then, we can formulate the following theorem:

Theorem 1. With an accuracy up to infinitesimal of the second order of the fraction $\left[\frac{V\left(x,z\left(x,z_0\right)\right)}{U\left(z\left(x,z_0\right)\right)}\right]^2$ the function $E_V\left(L,0;z_0\right)$ with $z_0 \in \left[0, Z\right)$ can be presented as

$$E_V\left(L,0;z_0\right) = -\frac{2\cdot\pi\cdot e^2}{\sqrt{m\cdot\left(\omega^2\left(z_0\right)+\nu^2_{\text{effective}}\right)}} \times \\ \times \int_{\Gamma_{OA}} \frac{V\left(x,z\left(x,z_0\right)\right)dx}{\sqrt{m\cdot\left(\omega^2+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U\left(z\left(x,z_0\right)\right)\right]}}. \qquad (13)$$

So, in the case (6), the initial problem (1)-(4) is presented as the following problem of integral geometry: it is required to restore the function $V\left(x,z\right)$ by its integrals by the series of curves

$$\int_{\Gamma_{OA}} K\left(z\left(x,z_0\right)\right)\cdot V\left(x,z\left(x,z_0\right)\right)dx = f\left(L,z_0\right), \qquad (14)$$

where

$$K\left(z\left(x,z_0\right)\right) \overset{def}{\equiv} \frac{1}{\sqrt{\Omega\left[U\left(z\left(x,z_0\right)\right)\right]}}, \qquad (15)$$

$$f\left(L,z_0\right) \overset{def}{\equiv} -\frac{\sqrt{m\cdot\left(\omega^2\left(z_0\right)+\nu^2_{\text{effective}}\right)}\cdot E_V\left(L,0;z_0\right)}{2\cdot\pi\cdot e^2}. \qquad (16)$$

V. SOLVING THE OBTAINED PROBLEM OF INTEGRAL GEOMETRY

The study of the problem (14)-(16) of integral geometry is done in two stages: first, we formulate and solve the auxiliary problem of integral geometry, where it is required to restore the function $V\left(x,z\right)$ by the measurement of intensity of electric field $E\left(x,z;\omega\right)$, $\omega \in \left(0,\omega_0\right)$, $\omega_0 \in \left(0,\omega_{\max}\right)$ only at one point; secondly, based on the data what we get in the first problem we study the general problem of integral geometry, where it is required to restore the function

$V(x,z)$ by measurement of intensity of electric field $E(x,z;\omega),\ \ \omega\in(0,\omega_j),\ \ \omega_j\in(0,\omega_{\max}),\ \ \forall j=\overline{1,n}$ by measurements at $(n+1)$ points.

A. The auxiliary problem and its investigation

Supposing that we know values $E(\omega)\overset{def}{\equiv}E(x,z;\omega)\big|_{(x,z)=(L,0)}$, for $\forall\omega\in(0,\omega_0)$, $\omega_0\in(0,\omega_{\max})$, from the problem (14)-(16) it is required to define the function

$$\varphi(z)=\begin{cases}0,\ z\in[0,Z);\\ \varphi(z)\in C[Z,+\infty),\ z\in[Z,+\infty),\end{cases}\quad(17)$$

which, then, means the function $V(x,z)\overset{def}{\equiv}\varphi(z)\cdot x.$

Theorem 2. There is no more than one function $\varphi(z)$ of the kind (17), which corresponds to each given first part $f(L,z_0)$ of the kind (16) of the equation (14).

Proof of the Theorem 2. From (12) and from the fact that

$$\alpha(z,z_0)\overset{def}{\equiv}\sqrt{\frac{m\cdot\left(\omega^2(z_0)+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U(z_0)\right]}{4\cdot\pi\cdot e^2}}\times$$
$$\times\int_z^{z_0}\frac{d\xi}{\sqrt{U(z_0)-U(\xi)}}\in D^2[Z,+\infty),$$

we can claim

$$\int_0^{z_0}\alpha'_z(z,z_0)\cdot\varphi(z)\,dz=\frac{f(L,z_0)}{L}.\quad(18)$$

Since

$$\alpha'_z(z,z_0)=\sqrt{\frac{m\cdot\left(\omega^2(z_0)+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U(z_0)\right]}{\pi\cdot e^2\cdot U'_z(z)\big|_{z=z_0}}}\times$$
$$\times\frac{2\cdot(z_0-z)\cdot\beta'_z(z,z_0)-\beta(z,z_0)}{2\cdot\sqrt{z_0-z}},$$

where

$$\beta(z,z_0)\overset{def}{\equiv}\frac{\alpha(z,z_0)}{\sqrt{(z_0-z)}}\times$$
$$\times\sqrt{\frac{\pi\cdot e^2\cdot U(z_0)}{m\cdot\left(\omega^2(z_0)+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U(z_0)\right]}},\quad(19)$$

Then equation (18) takes the following form:

$$\int_0^{z_0}\frac{\tilde{K}(z,z_0)}{\sqrt{z_0-z}}\cdot\varphi(z)\,dz=F(z_0),\quad(20)$$

where the core $\tilde{K}(z,z_0)$ and the right part $F(z_0)$ are defined by formulas

$$\tilde{K}(z,z_0)\overset{def}{\equiv}2(z_0-z)\beta'_z(z,z_0)-\beta(z,z_0);$$
$$F(z_0)\overset{def}{\equiv}\frac{2}{L}\sqrt{\frac{\pi e^2U'_z(z)\big|_{z=z_0}}{m\left(\omega^2(z_0)+\nu^2_{\text{effective}}\right)\Omega\left[U(z_0)\right]}}f(L,z_0).\quad(21)$$

Due to arbitrariness of the point $z_0\in[Z,+\infty)$, the obtained integral equation (20) is the first kind Volterra integral equation with a weak singularity. Consequently, in the space $C[Z,+\infty)$ of continuous function, the problem (20), (21) can have only one solution ([21]). With that, the proof of the Theorem 2 is finished.

Remark. The Theorem 2, which was just proved, does not give an answer to the question of existence of solution of the problem (20), (21), so, consequently, also to the above mentioned auxiliary problem of integral geometry, too. Any condition, which guarantees smoothness of the function $F(z_0)$ from (21) is the sufficient condition for solution of the auxiliary problem of integral geometry.

B. The original general problem and its investigation

Supposing that for $\forall j=\overline{0,n}$ we know values $E_j(\omega)\overset{def}{\equiv}E(x,z;\omega)\big|_{(x,z)=(L_j,0)}$ for $\forall\omega\in(0,\omega_j)$, $\omega_j\in(0,\omega_{\max})$, it is required from the problem (14)-(16) to define the function

$$\varphi_j(z)=\begin{cases}0,\ z\in[0,Z);\\ \varphi_j(z)\in C[Z,+\infty),\ z\in[Z,+\infty),\end{cases}\quad(22)$$

and, by that, the function $V(x,z)$.

Theorem 3. There is no more than one collection of functions $\{\varphi_j(z)\}_{j=\overline{1,n}}$ of the kind (22), corresponding to each set of the right-hand side $\{f_j(L_j,z_0)\}_{j=\overline{0,n}}$ of the kind (22) of the equation (14).

Proof of the Theorem 3. We consider a randomly chosen point L_k from the given set $\{L_j\}_{j=\overline{0,n}}$ and consider $V(x,z)$ in the equation (14) of integral geometry:

$$\int_0^{L_k}\left\{\sum_{i=0}^{n}\varphi_i\left(z_k\left(x,z_0\right)\right)\cdot x^i\right\}dx=$$
$$\int_0^{z_0}\left[\alpha_k\left(z,z_0\right)\right]'_z\cdot\sum_{i=0}^{n}\left\{\left(\alpha_k\left(z,z_0\right)+\frac{L_k}{2}\right)^i-\right.\qquad(23)$$
$$\left.-\left(\alpha_k\left(z,z_0\right)-\frac{L_k}{2}\right)^i\right\}\cdot\varphi_i\left(z\right)dz=f_k\left(L_k,z_0\right),$$

where $\alpha_k\left(z,z_0\right)\overset{def}{\equiv}\alpha\left(z\left(L_k,z_0\right),z_0\right)$.

Applying to the equation (23) the method analogical to the one applied at the first stage, while we were formulating the formulas (18)-(21), we get the following integral equation:

$$\int_0^{z_0}\sum_{i=0}^{n}\frac{L_k^i\cdot\tilde{K}_{k,i}\left(z,z_0\right)}{2^{i-1}\cdot\sqrt{z_0-z}}\cdot\varphi_i\left(z\right)dz=F_k\left(z_0\right),\quad(24)$$

where we mark

$$\left.\begin{aligned}&\tilde{K}_{k,i}\left(z,z_0\right)\overset{def}{\equiv}\tilde{K}^i\left(z\left(L_k,z_0\right),z_0\right);\\&F_k\left(z_0\right)\overset{def}{\equiv}\frac{2}{L_k}\cdot\sqrt{\frac{\pi\cdot e^2\cdot U'_z\left(z\right)\big|_{z=z_0}}{m\cdot\left(\omega^2\left(z_0\right)+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U\left(z_0\right)\right]}}.\end{aligned}\right\}$$

We introduce the following marking:

- $M\overset{def}{\equiv}\left\{\dfrac{L_i^j}{2^{j-1}}\right\}_{i=\overline{0,n}}^{j=\overline{0,n}}$ is the quadratic matrix of size $\left(n+1\right)\times\left(n+1\right)$;
- $\tilde{K}\overset{def}{\equiv}\left\{\tilde{K}_{i,j}\right\}_{i=\overline{0,n}}^{j=\overline{0,n}}$ is the quadratic matrix of size $\left(n+1\right)\times\left(n+1\right)$, constituents of which are calculated by the formula
$$\tilde{K}_{i,j}=\left(2\cdot\left(z_0-z\left(L_j,z_0\right)\right)\cdot\beta'_z\left(z\left(L_j,z_0\right),z_0\right)-\beta\left(z\left(L_j,z_0\right),z_0\right)\right)^i,$$
where the function $\beta\left(z\left(L_j,z_0\right),z_0\right)$ is defined by the formula (19);
- I is an identity matric of size $\left(n+1\right)\times\left(n+1\right)$;
- $\Phi\overset{def}{\equiv}\left\{\varphi_i\right\}_{i=\overline{0,n}}$ is a column-vector of size $\left(n+1\right)\times1$, constituents of which are sought-for functions of the equation (24) (consequently, of the original general problem of integral geometry);
- $F\overset{def}{\equiv}\left\{F_i\right\}_{i=\overline{0,n}}$ is the column-vector of size $\left(n+1\right)\times1$, constituents of which are calculated by the formula

$$F_j\overset{def}{\equiv}\frac{2}{L_j}\cdot\sqrt{\frac{\pi\cdot e^2\cdot f^2\left(L_j,z_0\right)\cdot U'_z\left(z\right)\big|_{z=z_0}}{m\cdot\left(\omega^2\left(z_0\right)+\nu^2_{\text{effective}}\right)\cdot\Omega\left[U\left(z_0\right)\right]}},$$

where $f\left(L_j,z_0\right)$ is a function in the form (16).

The above stated lets rewrite the integral equation (24) in a matrix form:

$$M\int_0^{z_0}\frac{\tilde{K}I\Phi\left(z,z_0\right)}{\sqrt{z_0-z}}dz=F\left(z_0\right).\quad(25)$$

Due to arbitrariness of the point $z_0\in\left[Z,+\infty\right)$ obtained matrix integral equation (25) is the first kind Volterra equation with a weak singularity. Therefore, in the space $C\left[Z,+\infty\right)$ of continuous functions this equation can have unique solution. Also, the remark made above at the first stage remains correct also for the matrix integral equation (25): guaranteeing smoothness of the right side of the F equation (25) guarantees the existence of its solution, so, it also guarantees the existence of the solution of the initial problem of integral geometry. The proof of the Theorem 3 is finished.

Solution of the Abel's integral equation (25) is the following vector-function (for instance, see [29]):

$$\tilde{K}I\Phi\left(z,z_0\right)\big|_{z=z_0}=\frac{1}{\pi}\cdot\frac{d}{dz_0}\int_0^{z_0}\frac{M^{-1}F\left(z,z_0\right)}{\sqrt{z_0-z}}dz.$$

From here follows the following system of linear algebraic equations with respect to sought-for functions $\varphi_i\left(z\right),\ i=\overline{0,n}$ at each point $z_0\in\left[Z,+\infty\right)$:

$$\sum_{j=0}^{n}\tilde{K}_{i,j}\left(z,z_0\right)\big|_{z=z_0}\cdot\varphi_j\left(z_0\right)=$$
$$\frac{1}{\pi}\cdot\frac{d}{dz_0}\int_0^{z_0}\frac{\left(M^{-1}F\left(z,z_0\right)\right)_i}{\sqrt{z_0-z}}dz,\ i=\overline{0,n}.\quad(26)$$

Having solved this system by some kind of direct methods, for $\forall i=\overline{0,n}$ we can find

$$\varphi_i\left(z_0\right)=\frac{1}{\pi}\cdot\left(\tilde{K}\left(z,z_0\right)\big|_{z=z_0}\cdot\frac{d}{dz_0}\int_0^{z_0}\frac{M^{-1}F\left(z,z_0\right)}{\sqrt{z_0-z}}dz\right)_i^{-1}.$$

However, it will be better to solve the system (26) by Tikhonov's regularization method for finding its stable solution (for instance, see [30], [31]).

VI. CONCLUSION

In the present work, we prove the existence and uniqueness of the solution of the inverse problem (1)-(4) as well as we propose the analytical method permitting: firstly, to reduce it to the problem of integral geometry, and thereupon, having applied the adjusted variant of the Lavrentiev's theorem, to reduce the obtained problem of integral geometry to the first kind matrix integral equation of Volterra type with a weak singularity.

Let's note that the considered problem arises, generally, at study of the following problems (for instance, see [22]-[28] and respective references given in these): propagation of various low-frequency electromagnetic waves in the ionosphere, exosphere and adjacent to its regions of interplanetary space; propagation of radio waves in the ionosphere, i.e. in the upper layers of the Earth's atmosphere; propagation of radio waves of cosmic origin in the solar atmosphere, in the nebulae as well as in the interstellar and interplanetary spaces; propagation of radio waves at laser ranging of the Sun, the Moon and some planets as well as in case of communication with the distant artificial Earth satellites and space rockets; propagation of low-frequency magneto-hydrodynamic and acoustic waves in space environment; propagation of plasma waves both on the ionosphere and the solar corona; propagation of various types of electromagnetic waves in plasma created in vitro (i.e. at the laboratory conditions) at study of gaseous discharge as well as in installations meant for study of controlled thermonuclear reactions, etc.

It is appropriate also to mention here that by now the comprehension and the solid knowledge of the ionosphere have in many respects well-composed and completed character, and therefore, many sections of the discipline about the Earth's atmosphere is unlikely to undergo a change in the future. However, at present there are variety of investigated issues having the corresponding theoretical foundations only in a state of becoming: such fundamental issues as the formation of the ionosphere, the processes in the transition region between the ionosphere in the interplanetary medium, cloud formation mechanisms, wave excitation mechanisms, etc. have not yet been resolved in a satisfactory extent, and even there are variety of gaps and contradictions in the theories constructed for them.

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