UNCERTAIN PROBABILITIES
Nenoteiktās varbūtības

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Abstract
The uncertainty of probabilistic evaluations results from the lack of sufficient information and/or knowledge underlying those random events. Uncertainty representation in the form of second order probability distribution or interval evaluations does not cause any objections from the theoretical point of view. On the other hand, what is worthy in the second order probabilities is that they allow one to model a real uncertainty of subjective probabilistic evaluations resulting from the lack of information and/or knowledge. Processing of uncertain information regarding probabilistic evaluations can help make a validated decision about the collection of additional information aimed to remove completely or to reduce the existing uncertainty.
Keywords: *probabilistic evaluations, uncertainty, decision making.*

Sources of occurrence of uncertain probabilistic evaluations

Whenever practical applications of probability theory are considered, it is explicitly assumed that all probabilistic evaluations are of deterministic nature. Strictly speaking, one can speak correctly about the absolute validity of probabilistic evaluations only within the classical approach to probability evaluation. In turn, the validity of probabilistic evaluations within the frequency approach greatly depends on the volume and validity of the initial statistical data. As regards subjective probabilistic evaluations, the theory of subjective probabilities imposes a strict requirement: the expert has to assign unambiguous point evaluations to the probabilities of random events or variables being evaluated. The theory of subjective probabilities is entirely based on that requirement. It simply forbids any uncertainty in probabilistic evaluations.

Let us consider the possibilities of practical implementation of this requirement in more detail. As is generally known, one of the underlying postulates of the general theory of measurement is formulated as the necessity to correctly account the measurement errors. Every measurement of physical values can be performed within the accuracy ensured by measuring equipment and conditions of measurement. Hence, the results of any measurement can always be represented in the form $A \pm \varepsilon$, where A is the result of measurement but ε is possible measurement error. In essence, that form represents a confidence interval within which there is for guarantee situated the real meaning of the measured value.

The occurrence sources of probabilistic evaluation uncertainties are described in numerous literature [1-6]. The main source of potential uncertainty is the uncertainty regarding the underlying events, facts, statements and hypotheses. The theory of subjective probabilities is based on the statement that any subjective probability assignment is made on the basis of all the information available. Frequently, that fact is explicitly emphasised by denoting the subjective probability of event ***e*** as $p(e\ /\ \zeta)$ where ζ is the information, on the basis of which the evaluation of $p(e\ )$ was performed. From this it can be concluded correctly that the subjective probabilistic evaluation is in essence an evaluation of the conditional probability. When making subjective evaluation of probabilities, the expert explicitly or implicitly takes into account the variety of conditions ζ. If the conditions are uncertain for the expert, it is quite natural that it would be difficult for him to produce point-valued probabilities required. The evaluation becomes uncertain for him over the whole set of uncertain conditions, ζ.

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There also exists another important source of uncertainties or ambiguities in assigning subjective probability values. The reason for that could be mental limitations of ability of individuals to assign point-valued probabilities under the existing state of knowledge. Frequently, individuals cannot distinguish separate gradations of probabilities, even if there is enough information. Numerous studies have shown that certain extent of uncertainty is an inherent attribute of human thinking. The inability to distinguish and interpret sufficiently close probability values is akin to the inability of human beings to distinguish close colour nuances.

Second order probabilities

Current uncertainties regarding the values of probabilities can conveniently be modelled by belief networks. Fig. 1 represents an ordinary belief network [5].

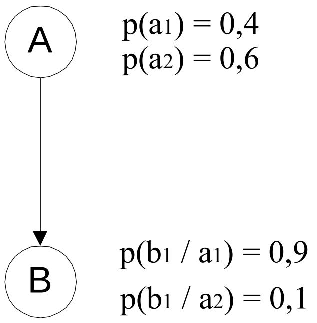

Fig. 1. Belief network that models the relationship between the disease and symptom

Node A represents two random events: a_1 –presence at the patient of the certain disease, a_2 – absence at the patient of this disease. Node B represents two random events: b_1 – presence at the patient of the certain symptom, b_2 – absence at the patient of this symptom. Unconditional probabilities of event a_1 and a_2 as well as conditional probabilities of event b_1 provided event a_1 and event a_2, are specified. Conditional probabilities $p(b_2 / a_1)$ and $p(b_2 / a_2)$ are not of interest to us in the example under consideration, so their values are not set.

Let us first assume that all the probability values are determined on the basis of extensive statistical data, so their validity is high. If symptom b_1 is observed at the patient, the posterior conditional probability of disease a_1 can easily be calculated. By using Bayes' formula we obtain

$$p(a_1 / b_1) = p(b_1 / a_1)\, p(a_1) / (p(b_1 / a_1)\, p(a_1) + p(b_1 / a_2)\, p(a_2)) =$$
$$= 0{,}9*0{,}4 / (0{,}9*0{,}4 + 0{,}1*0{,}6) \approx 0{,}86.$$

On the basis of this probability value, the physician can make a decision about the method of treatment.

Let us now assume that there is no statistical data to determine the objective values of probabilities, so the expert is asked to assign the probabilities. The expert has a large experience in the area under consideration. He assigns with a large extent of confidence the same value of probabilities $p(b_1 / a_1)$ and $p(b_1 / a_2)$ that were determined previously on the basis of statistical data. The expert, however, finds difficulty in assigning point values of probability $p(a_1)$. Considering certain values of probability $p(a_1)$ as random events c_1, c_2, c_3, he has assigned the probabilities of those events as follows:

$$p(c_1) = p(p(a_1) = 0{,}1) = 0{,}2;$$

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$$p(c_2) = p(p(a_1) = 0,4) = 0,6;$$
$$p(c_3) = p(p(a_1) = 0,7) = 0,2.$$

The existing state of information is modelled using the belief network shown in Fig.2.

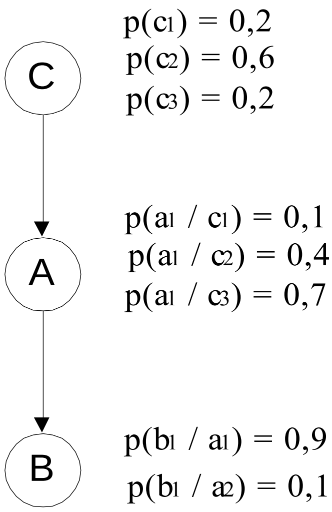

Fig. 2. Belief network modelling the uncertainty of probability values $p(a_1)$

The only way to determine point values of probability $p(a_1)$ is to calculate its expected value over the whole available set of random values:

$$E(p(a_1)) = \sum_{i=1}^{n} p(a_1 / c_i)\, p(c_i) = 0,1*0,2 + 0,4*0,6 + 0,7*0,2 = 0,40.$$

The calculated expected value $E(p(a_1))$ is exactly equal to the value of probability $p(a_1)$ previously determined by using valid statistical data. If the posterior probability of disease a_1 given symptom b_1 is determined, the same value of that probability will be obtained, i.e. 0,86. Although being formally similar, these evaluations represent quite different states of prior information. The first evaluation obtained on the basis of objective initial information has a large confidence degree. In other words, if the physician has such evaluation at his disposal, he can make a decision concerning the method of treatment in full confidence that this evaluation is exactly equal to the probability of presence of this disease at the patient. In the second instance the matter is far from being so successful. From uncertainty of expert's evaluation actual value $p(a_1)$ can be far from expected value. Using this example the following important conclusion can be drawn. When uncertain probabilistic evaluations are employed, the transition to the mathematical expectation does not reduce the initial uncertainty. That uncertainty is implicitly included in further calculations and leads to the implicit uncertainty of the results and consequences of the actions undertaken. An explicit account of that uncertainty by means of calculating the interval of possible values of the resulting probability can help further analyse the uncertainties in the following way. If the user is satisfied with that interval of probability uncertainty in the context of the problem of interest, he may use the expected value as a point value of the corresponding probability. If the uncertainty is large, a decision to collect additional information can be made. In other words, the correct analysis of uncertainties of probabilistic evaluations cannot raise the validity of the final results. It, however, makes the basis of evaluation of the suitability extent of the initial information.

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Interval probabilities and multiple probability distributions

Specification of probabilistic evaluations in the form of intervals of possible values represents essential lack of information and/or knowledge on the subject domain. If the expert is not able to unambiguously evaluate probabilities of events and even assign the probability distribution in the set of uncertain probabilistic evaluations, he may simply set intervals of possible values of the required probabilistic evaluations. It can easily be seen that interval evaluations of probabilities lead to a set of probability distributions compatible with the existing state of information. Let us consider an evident example illustrating how the lack of the initial information is translated into the essential uncertainty of the results [7]. A hypothetical sensor produces indication of the temperature of the process of production. There are two temperature gradations: high (HT) and low (LT). When the temperature is high, the lamp is red (RC). If the temperature is low, it has a blue colour (BC). Unfortunately, sensor's thermometer is a very fragile device and can be broken even at a slight shaking. The probability that it is broken is 20% at any moment of time. When the thermometer is broken, the sensor indication is not related to the real temperature of the process. Imagine a new technician is observing that the lamp is blue. What is the real temperature at these conditions? How it is possible to evaluate the probability that the temperature is really high or low?

Cartesian product defines a common space of all possible scenarios for that situation

$$\Omega = S \times T \times Q,$$

where $S = \{BC, RC\}$ – a set of sensor indication states;

$T = \{HT, LT\}$ – a set of temperature states of the process;

$Q = \{WS, NWS\}$ – a set of sensor thermometer states: WS – working state; NWS – non-working state (the thermometer is broken) .

All the scenarios are shown in Table 1. Each scenario is denoted by letter a, b, … , and h.

Let us show that it is not possible to perform common probabilistic analysis under this state of information. From Table 1 it follows that these limitations are valid:

$$p\,(WS) = p(a) + p(b) + p(c) + p(D) = 0{,}8; \quad (1)$$
$$p\,(NWS) = p(e) + p(f) + p(g) + p(h) = 0{,}2. \quad (2)$$

Table 1.

Set of possible scenarios of states of a hypothetical sensor

State of sensor's thermometer	Colour of sensor			
	Blue colour (Blue)		Red colour (Red)	
	Temperature			
	HT (High)	LT (Low)	HT (High)	LT (Low)
WS (Working)	a	b	c	d
NWS (Non-working)	e	f	g	h

When the thermometer of the sensor is in operation (WS), the sensor has red colour (RC) at the high temperature (HT) and blue colour (BC) at the low temperature (LT). From this it follows that

$$p(a) = p(d) = 0.$$

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If the sensor's thermometer is in non-working state (NWS), sensor indication (RC or BC) is not related to the temperature value: HT or LT. The state of the sensor can be represented as

$$p(BC / NWS, HT) = p(BC / NWS, LT).$$

Hence,

$$p(e) / (p(e) + p(g)) = p(f) / (p(f) + p(h)).$$

Let us denote the probability that the sensor has a blue colour provided that the thermometer is broken as

$$\alpha = p(BC, NWS) = (p(e) + p(f)) / ((p(e) + p(f) + p(g) + p(h))$$

and the posterior probability that the temperature is low as

$$\beta = p(LT) = p(b) + p(f) + p(h).$$

As the state of the thermometer (WS or NWS) depends on another effects and is not related to the temperature (HT or LT), we have

$$p(WS / HT) = p(WS),$$

hence,

$$(p(a) + p(c)) / (p(a) + p(c) + p(e) + p(g)) = 0{,}8.$$

Using limitations (1) and (2), and all the preceding statements, Table 1 can be represented as follows (Table 2).

Table 2.

Probability distribution of scenarios in the task of hypothetical sensor

State of sensor's thermometer	**Colour of sensor**			
	BC (Blue)		RC (Red)	
	Temperature			
	HT (High)	LT (Low)	HT (High)	LT (Low)
WS (Working)	0	$0{,}8\beta$	$0{,}8(1 - \beta)$	0
NWS (Non-working)	$0{,}2(1 - \beta)\alpha$	$0{,}2\beta\alpha$	$0{,}2(1-\beta)(1-\alpha)$	$0{,}2\beta(1 - \alpha)$

The existing limitations (1) and (2) do not enable one to unambiguously determine values α and β. We are interested in knowing the probability that the temperature of the process is really low at the blue colour of the sensor and the existing state of information. Using the data of Table 2 we receive the following expression for the probability under consideration:

$$p(LT / BC) = (p(b)+p(f)) / (p(a)+p(b)+p(e)+p(f)) = (0{,}8 + 0{,}2\alpha)\beta / (0{,}8\beta + 0{,}2\alpha).$$

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Even if probability β is known, it is not clear which value must probability α have to unambiguously define p(LT / BC). The information available is not sufficient to solve the task stated by means of conventional apparatus of probability theory. It is only possible to set some intervals for values α and β.

To reduce uncertainty in the tasks of this kind, one can employ the principle of maximum entropy or a model of lower and upper probabilities.

The method of interval probabilistic evaluations proposed by Neapolitan [5] is also of interest. The method is worked out for the case when there is a system of *n* random events, and the probability values for those events are specified in the form of intervals. Let us first consider a case when the system includes two random events $E = e_1, e_2$. The probabilities of the events are:

$$p(e_1) \in [0;\ 0{,}5]; \qquad p(e_2) \in [0{,}5;\ 1].$$

Let us include random variables X_1 and X_2. in the consideration. X_1 represents possible values of probability $p(e_1)$ but X_2 represents possible values of probability $p(e_2)$. Variable X_1 may assume values in the interval [0; 0,5]. Since the values of probabilities $p(e_1)$ and $p(e_2)$ are connected by relationship $p(e_1) + p(e_2) = 1$, assigning of any value for X_1 unambiguously determines the corresponding value of X_2, and vice versa. Assigning the interval of possible values of the probability does not assume any probabilistic distribution in this interval. One can, however, suppose that possible values of probabilities have a uniform distribution in the interval. That assumption is not in contradiction to the initial conditions. Then, treating the expected values of probabilities $p(e_1)$ and $p(e_2)$ in the corresponding interval as point-valued probabilities, we have:

$$E(X_1) = E(p(e_1)) = \int_0^{0{,}5} x_1/(0{,}5 - 0)dx_1 = 0{,}25;$$
$$E(X_2) = E(p(e_2)) = \int_{0{,}5}^{1} x_1/(1 - 0{,}5)dx_1 = 0{,}75.$$

It should be noted that an assumption about the uniform distribution density of the evaluated variable in the interval of its determination is nothing but an assumption. It does not represent the real state of things but at the same time does not contradict evidently this state of things. Setting the probability distribution of the values of the random variable and setting an interval of its possible values are two different things that are not related to each other. The method suggested by Neapolitan is an attempt to at least consistently connect two different representations of uncertain information with each other.

Now consider the Neapolitan method in general form. A complete system of *n* random events is specified. The probabilities of the events are set in the form of intervals:

$$p_i \in [a_i;\ b_i],\ i = 1, \dots, n.$$

Denote a random variable representing the uniform distribution values of probability p_i in the i-th interval as X_i. The difficulty is that point values $E(p_i)$ must satisfy the requirement of connectivity $\sum_{i=1}^{n} E(p_i) = 1$. To solve the task formulated, the author proposes first to determine new intervals of values of the corresponding probabilities as follows:

$$x^*_2(x_1) = \max(a_1,\ 1 - x_1 - b_3 - b_4 - \dots - b_n);$$
$$x^{**}_2(x_1) = \min(b_2,\ 1 - x_1 - a_3 - a_4 - \dots - a_n);$$

$$x^*_3(x_1, x_2) = \max(a_3,\ 1 - x_1 - x_2 - b_4 - \dots - b_n);$$
$$x^{**}_3(x_1, x_2) = \min(b_3,\ 1 - x_1 - x_2 - a_4 - \dots - a_n);$$

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$$x^*_{n-1}(x_1, x_2, \dots, x_{n-2}) = \max(a_{n-1}, 1 - x_1 - x_2 - \dots - x_{n-2} - b_n);$$
$$x^{**}_{n-1}(x_1, x_2, \dots, x_{n-2}) = \min(b_{n-1}, 1 - x_1 - x_2 - \dots - x_{n-2} - a_n).$$

Then value $\mu_1(x_1)$ is calculated using expression that follows:

$$\mu_1(x_1) = \left(\int_{x^*2}^{x^{**}2} \int_{x^*3}^{x^{**}3} \dots \int_{x^*n-1}^{x^{**}n-1} dx_{n-1}dx_{n-2} \dots dx_2\right) / \left(\int_{a1}^{b1} \int_{x^*2}^{x^{**}2} \int_{x^*3}^{x^{**}3} \dots \int_{x^*n-1}^{x^{**}n-1} dx_{n-1}dx_{n-2}dx_1\right).$$

The expected value of probability p_1 is calculated as

$$E(p_1) = \int_{a1}^{b1} x_1\, \mu(x_1)dx_1.$$

Then all the calculations are repeated in the same way for probabilities $p_2, \dots, p_n$.
The values of integrals in the above expressions can be calculated by means of integer integration.

Example. Assume that a complete system of random events $E = e_1, e_2, e_3, e_4$ is set with the following interval values of probabilities:

$$p(e_1) \in [0,2; 0,5]; \quad p(e_2) \in [0,2; 0,4]; \quad p(e_3) \in [0,1; 0,6]; \quad p(e_4) \in [0; 0,5].$$

Applying the above-considered method we obtain these point values of probabilities:

$$E(p(e_1)) = 0,320; \quad E(p(e_2)) = 0,287; \quad E(p(e_3)) = 0,246; \quad E(p(e_4)) = 0,146.$$

The undoubted advantage of the method is that it automatically meets the requirement of connectivity $\sum_{i=1}^{n} p(e_i) = 1$. The shortcoming of this method is computational difficulties.

Uncertain probability evaluations: the pros and cons

The uncertainty of probabilistic evaluations results from the lack of sufficient information and/or knowledge underlying those random events. Uncertainty representation in the form of second order probability distribution or interval evaluations does not cause any objections from the theoretical point of view. However, due to uncertain probabilities, certain problems of conceptual nature arise. First, many theorists and practitioners have a sharply expressed aversion of the fact that probability values act as random events. Educated in the way of classic probability theory, certain scientists are in principle against second order probabilities. Many adherents of the subjective probability theory are also against second order probabilities as these probabilities are simply forbidden by that theory. There exists another conceptual problem related to second order probabilities that consists in that, if second order probabilities are incorporated, then there is no any principal obstacle to incorporate probabilities of the third order and higher. In the limit case it could lead to infinite hierarchy of probabilities. This is the problem, which many scientists who work in the area of probability are anxious about. Real life, however, shows, that even in the most complicated situations it is quite enough to work with probabilities of the second order, or in the extreme case, of the third order. That is why the problem of the infinite hierarchy seems to be far-fetched.

On the other hand, what is worthy in the second order probabilities is that they allow one to model a real uncertainty of subjective probabilistic evaluations resulting from the lack of information and/or knowledge. In classic probability theory, the problem of probability

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uncertainty simply does not exist as probability evaluations are based on extensive fact material and strict logical argumentation.

Based on the above short examination of uncertainties of the probability evaluations, the following general conclusions can be drawn.

1. Application of the second order probabilities and interval values makes it possible to model natural uncertainties of expert judgements regarding the evaluations assigned. These uncertainties are the result of insufficient initial information and/or knowledge.

2. The source of uncertainty of probabilistic evaluations is uncertainties underlying the conditions and limitations of human mental activities.

3. Various methods to manage uncertain probabilistic evaluations are developed. Each of the methods possesses both advantages and shortcomings.

4. Processing of uncertain information regarding probabilistic evaluations can help make a validated decision about the collection of additional information aimed to remove completely or to reduce the existing uncertainty.

5. Probability theory is successfully employed to cope with the uncertainties of the surrounding world provided that a whole series of fairly strict conditions are satisfied. The incorporation of uncertain probabilistic evaluations allows one to broaden the existing boundaries of probability theory application since the world surrounding us is too complicated to be described successfully by means of classical probability only.

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